MAT1344 22/1/2021

* office hr after class.
* If you're taking this for credit please email me:
  1. between 1 sentence & 1 paragraph:
     what was unclear?
     on what should I expand?
     request?
  2. between 1 paragraph & 1 page: MATH!
     work out details of something?
     from these 2 weeks.

An excellent reference on Lie groups & their actions:
John Lee "Intro to smooth manifolds"
chapters 7, 8, ~~9~~, 20, 21.

Hamiltonian G-space :

$$G \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$$

Lie gp

sympl mfld

(G-equivariant) momentum map.

$\mathfrak{g} = \mathrm{Lie}(G) := T_1 G$.

{\mathfrak g}

Hamilton's equation $\forall \xi \in \mathfrak{g}$

$$\boxed{d\mu^{\xi} = -\xi_M \lrcorner\, \omega}$$

for $\mu^{\xi} := \langle \mu, \xi \rangle : M \longrightarrow \mathbb{R}$

Example. $G = (S^1)^n \circlearrowright \mathbb{C}^n = \mathbb{R}^{2n}$

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j = \sum_{j=1}^{n} r_j\, dr_j \wedge d\theta_j$$

$$(a_1, \cdots, a_n) \cdot (z_1, \cdots, z_n) := (a_1 z_1, \cdots\cdots, a_n z_n)$$

$$\mathbb{R}^n \cong \mathrm{Lie}(S^1)^n \longrightarrow \mathfrak{X}(\mathbb{C}^n)$$

$\xi \mapsto \xi_M$ vector fields

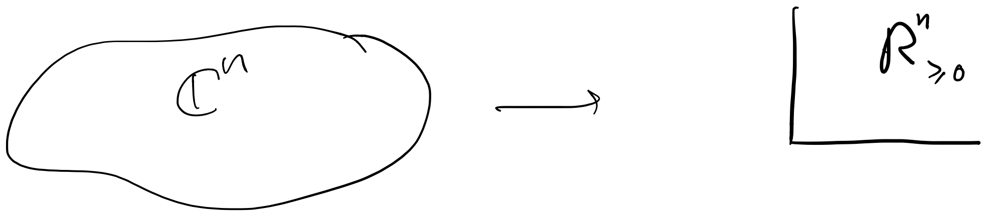
$$e_j \longmapsto \frac{\partial}{\partial \theta_j} = x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}$$

$$\mu : \mathbb{C}^n \longrightarrow (\mathrm{Lie}(S^1)^n)^* \cong \mathbb{R}^n$$

$\mu = (\mu_1, \cdots, \mu_n)$

$$d\mu_j = -\frac{\partial}{\partial \theta_j} \omega = r_j\, dr_j$$

$$\therefore \mu(z_1, \cdots, z_n) = \left(\frac{|z_1|^2}{2}, \cdots, \frac{|z_n|^2}{2}\right) + \text{constant}$$



Lie algebras

A **Lie algebra** is a vector space V **equipped** with an antisymmetric bilinear map

$$[\ ,\ ] : V \times V \longrightarrow V$$

that satisfies Jacobi's identity:

$$\forall\, u, v, w \qquad [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

Example $V = \mathbb{R}^{n\times n} = n\times n$ matrices

$$[A,B] = AB - BA \quad \text{for } A, B \in \mathbb{R}^{n\times n}$$

Example $\mathfrak{X}(M) := \{\text{vector fields on } M\}$.

is a Lie algebra with

$$[X,Y]f = XYf - YXf \quad \forall f \in C^\infty(M)$$

Let G be a Lie group. & $a, b \in G$.

$L_a : G \to G \qquad g \mapsto ag$

$R_b : G \to G \qquad g \mapsto gb$

$(L_a)_* : TG \longrightarrow TG$

write as $\qquad \xi \longmapsto a\xi$

Here if $\xi \in T_g G$ is represented by $\gamma : \mathbb{R} \to G$, $\gamma(0) = g$

~~and~~ then $a\xi \in T_{ag} G$ is represented by $t \mapsto a\gamma(t)$

A vector field X on G is left invariant if

$\forall a, g \in G \qquad a \cdot (X|_g) = X|_{a \cdot g}$

This holds $\Leftrightarrow$ $\forall f \in C^\infty(G)$

$$(Xf)(ag) = X(g \mapsto f(ag))$$

$\therefore$ If X, Y are left invariant v. fields,

so is $[X, Y]$.

(this generalizes to vector fields on any mfld w/ G-action)

$$\mathfrak{g} := T_1 G \longleftrightarrow \left\{ \begin{array}{c} \text{left invariant} \\ \text{vector fields on } G \end{array} \right\}$$

$$\xi \longleftrightarrow X \quad \text{s.t.} \quad X\big|_g = g \cdot \xi$$

Define $[\,,\,]$ on $\mathfrak{g}$ through this bijection!

Lemma

For every homomorphism of Lie groups

$$\psi : H \longrightarrow G \,,$$

$$\begin{array}{ccccc} d\psi\big|_1 & : & T_1 H & \longrightarrow & T_1 G \\ & & \| & & \| \\ & & \mathrm{Lie}(H) & & \mathrm{Lie}(G) \\ & & \overset{!!}{\|} & & \overset{!!}{\|} \\ & & \mathfrak{h} & & \mathfrak{g} \end{array}$$

is a homomorphism of Lie algebras.

$\boxed{\text{Isotropy representation}}$

Let $G \circlearrowright N$

Let $x \in N$ be a fixed point. i.e.

$\forall g \in G \quad g \cdot x = x$

i.e. $G \cdot x = \{x\}$

i.e. $Stab(x) = G$.

Then $\forall a \in G$ $\quad a : N \longrightarrow N$

$x \longmapsto x$

induces $a_* : T_x N \longrightarrow T_x N$

$\quad\quad \| $

$\quad da|_x$

$a \mapsto a_*$ is a linear representation

$G \circlearrowright T_x N$.

$=:$ "the isotropy representation".

Note:

if $G \circlearrowright (M, \omega)$ preserves ω

i.e. $\forall a \in G \quad a^* \omega = \omega$

Then $G \circlearrowright T_x M$ preserves $\omega|_x$.

$\boxed{\text{Coadjoint action}}$ $\quad G \circlearrowright \mathfrak{g}^*$

start with $G \circlearrowright G = M$ by conjugation:

$$a: \quad g \longmapsto a g a^{-1} .$$

It preserves 1.

Its linearization: $Ad: G \circlearrowright T_1 G = \mathfrak{g}$

is called the <u>adjoint representation</u>

The dual representation
is called the <u>coadjoint representation</u>.

$$Ad^*: G \circlearrowright \mathfrak{g}^*$$

$$\forall a \in G \qquad Ad(a): \mathfrak{g} \to \mathfrak{g}$$
$$Ad^*(a): \mathfrak{g}^* \to \mathfrak{g}^*$$

$$\langle Ad^*(a)\varphi, \xi \rangle = \langle \varphi, Ad(a^{-1})(\xi) \rangle \quad \forall \xi \in \mathfrak{g}$$
$$\forall \varphi \in \mathfrak{g}^*$$

We require a momentum map
$$\mu: M \longrightarrow \mathfrak{g}^*$$
to be <u>equivariant</u> wrt the given G action on M
& the coadjoint action on $\mathfrak{g}^*$.

Definition.

The <u>coadjoint orbits</u> of G are the orbits of $Ad^*: G \circlearrowright \mathfrak{g}^*$.

Let $M \subseteq \mathfrak{g}^*$ be a coadjoint orbit of G.

- M $\boxed{\text{is}}$ a manifold.

  M is a weakly embedded submfd of $\mathfrak{g}^*$

  ($\Rightarrow$ $\exists!$ mfd str on M s.t. $M \hookrightarrow G$ is an immersion)

- M $\boxed{\text{is}}$ a sympl. mfld.

  we <u>equip</u> M with the "Kirillov-Kostant-Souriau sympl. form"

- $G \circlearrowright M$ transitively

- The inclusion map $\mu: M \hookrightarrow \mathfrak{g}^*$ is a momentum map.

office hours

$$\psi : M \longrightarrow N$$

smooth map of manifolds.

$x \in M \qquad y = \psi(x) \in N$

$T_x M$ = "linear approximation of M at x"

$T_y N$ = " ——"—— N at y"

$$\psi_* = d\psi\Big|_x : T_x M \longrightarrow T_y N$$

= "linear approximation of ψ at x".

In coordinates:

if $\psi : (u_1, \ldots, u_m) \longmapsto (v_1, \ldots, v_n)$

$$v_j = \psi_j(u_1, \ldots, u_m)$$

$x = 0$ $\qquad \psi(0) = 0$

$y = 0$ $\qquad$ Then $L := \psi_* : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

is repr. by the matrix $\left[\dfrac{\partial \psi_j}{\partial u_i}\right]$.

$$\left(\begin{array}{c} \Psi(u) = \Psi(0) + L(u) + \mathrm{Error}(u) \\ \text{and} \quad \dfrac{\|\mathrm{Error}(u)\|}{\|u\|} \xrightarrow[u \to 0]{} 0 \end{array} \right)$$

Let G be a compact connected Lie gp.

$G \subseteq G_{\mathbb{C}}$ its complexification

(the connected cplx Lie gp

s.t. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$)

$$\begin{array}{ccc} B & \subset & G_{\mathbb{C}} \\ \cup & & \cup \\ T & \subset & G \end{array}$$

Borel subgp.

maximal torus

Take $M := Ad^*(G) \cdot \lambda$ $\quad \lambda \in \mathfrak{g}^*$ s.t. $\mathrm{stab}(\lambda) = T$

$$G \longrightarrow M$$
$$a \longmapsto Ad^*(a)(\lambda)$$

induces $G/T \xrightarrow{\simeq} M$ sympl.

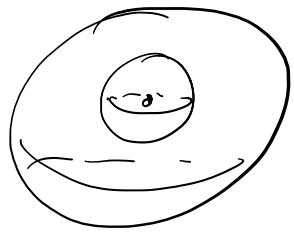
$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & G_{\mathbb{C}} \arrow[d] \\ M \cong G/T \arrow[r, "\simeq"] & G_{\mathbb{C}}/B \end{tikzcd}$$

complex mfld

$Ad^*: SU(2) \circlearrowright \mathbb{R}^3$

through $SU(2) \longrightarrow SO(3)$

orbits:



Kirillov-Kostant-Souriau:

$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{x^2+y^2+z^2}$$ on each sphere

Need to check.

$M = G/T$

Tangent: $\mathfrak{g}/\mathfrak{t}$.

$T \circlearrowright \mathfrak{g}$

$\underline{\text{f. pt set } \mathfrak{t}.}$

$T \circlearrowright \mathfrak{g}/\mathfrak{t}$ with trivial fixed subspace.

Construct coadjoint orbits
as Kähler reduction?
(of T^*G ?)

G compact $\circlearrowleft M$ $\qquad G \cdot x = \{x\}$

Then $\exists$ nd $\overset{G\text{-invt}}{U}$ of x in M

and a $\underbrace{G\text{-invt}}_{\text{wrt the isotr. repr.}}$ nd Ω of 0 in T_xM

can take this to be all of T_xM

and a G-equivariant diffeo

$$\begin{aligned} U &\longrightarrow \Omega \\ x &\longmapsto 0 \end{aligned}$$

Momentum map $\quad \mu : M \longrightarrow \mathfrak{g}^*$

$$\begin{aligned} M \times \mathfrak{g} &\longrightarrow \mathbb{R} \\ m, \xi &\longmapsto \mu^\xi(m) \end{aligned}$$

$$\begin{aligned} \mu^\# : \mathfrak{g} &\longrightarrow C^\infty(M) \\ \xi &\longmapsto \left(m \mapsto \mu^\xi(m)\right) \end{aligned}$$

If G is connected & μ satisfies Hamilton's equation:

μ is G-equivariant

$\iff$ $\mu^\#$ is a Lie algebra morphism

$$(\mathfrak{g}, [\,,\,]) \longmapsto \big(C^\infty(M), \{\,\}\big)$$

Poisson bracket