MAT1344 27/1/2021.

Welcome back!

practical.

- HW - reminder.

current topic:

Coadjoint orbits.

submanifolds

Note a manifold structure on a set N is determined by $\{ N \xrightarrow{C^\infty} \mathbb{R} \}$.

It is also determined by $\{ U \xrightarrow{C^\infty} N \}$.
$U \subset \mathbb{R}^n$ open, $n = 1, 2, 3, \dots$

(exercise)

Given a subset N of a manifold M, define:

$f : N \to \mathbb{R}$ is smooth $:\Longleftrightarrow$
for each point of N $\exists$ open nbhd U in M
and $\exists$ smooth $F : U \to \mathbb{R}$ s.t. $F|_{U \cap N} = f|_{U \cap N}$.

$p : U \to N$ is smooth $:\Longleftrightarrow$ the composition
$\cap$ open
$\mathbb{R}^n$, $n \in \mathbb{N}$

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[rr, bend right] & N \arrow[r, hook, "i"] & M \end{tikzcd}$$

is smooth.

Definition
A subset N of a mfld M is an embedded submanifold if there exists a (necessarily unique) manifold structure on N such that for every function $f : N \to \mathbb{R}$
f is smooth on N as a subset of M
iff f is smooth on N as a manifold.

Exercise show that this is equivalent to your favourite definition of embedded submanifold.

Definition
A subset N of a mfld M is a <u>weakly embedded submanifold</u> if

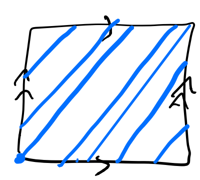
① there exists a (necessarily unique) manifold structure on N such that for every map $p: U \longrightarrow N$, $U \subset \mathbb{R}^n$ open, $n = 1, 2, 3, \ldots$

p is smooth to (N as a subset of) M
iff p is smooth to N as a manifold.

② wrt this manifold structure, $i : N \hookrightarrow M$ is an immersion.

Example $\quad M = \mathbb{R}^2/\mathbb{Z}^2 \qquad N = \{[t, \alpha t] \mid t \in \mathbb{R}\}$
for some $\alpha \notin \mathbb{Q}$.

$t \mapsto [t, \alpha t]$
is a diffeomorphism from $\mathbb{R}$ to N



when N is viewed as a weakly embedded submanifold.

Exercise: If N is embedded, then it is weakly embedded, with the same manifold structure.

"submanifold" means "embedded submanifold".

Sub Lie groups

a sub-Lie-group of a Lie group G is a subset H of G such that

- H is a subgroup
- H is a weakly embedded submanifold.

Equip H with this manifold structure.

Exercise: H is a Lie group.
with its induced mfld str & group str.

Exercise. Let $i : H \hookrightarrow G$ be the inclusion map of a Lie subgroup.
Then $di_1 = i_* : T_1 H \longrightarrow T_1 G$ identifies $\mathfrak{h}$

$$T_1 H = \mathrm{Lie}(H) = \mathfrak{h}, \qquad T_1 G = \mathrm{Lie}(G) = \mathfrak{g}$$

with a sub Lie algebra of $\mathfrak{g}$.

Moreover, with this identification,

$$\mathfrak{h} = \{ \xi \in \mathfrak{g} \mid \exp(t\xi) \in H \ \forall t \in \mathbb{R} \}$$

Thus, we obtain a map $\left\{\begin{matrix}\text{Lie subgroups}\\ H \subseteq G\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Lie subalgebras}\\ \mathfrak{h} \subseteq \mathfrak{g}\end{matrix}\right\}$.

- This map is onto.

Moreover, given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$,
it comes from
$H :=$ the group generated by $\exp(\xi)$ for $\xi \in \mathfrak{h}$.

- Two Lie subgroups have the same Lie algebra iff they have the same identity component.

Moreover: The identity component H_0 of a Lie subgroup H of G is open in H and is itself a Lie subgroup of G.

Recall: The identity component H_0 of H is the connected component of H wrt its topology as a manifold that contains the identity element 1.
It's an open & closed subset of H and it's a subgroup of H.
In fact, H_0 is generated by $\{\exp \xi \mid \xi \in \underset{\substack{\parallel \\ \mathrm{Lie}(H)}}{\mathfrak{h}}\}$.

Theorem. If G is a Lie group and $H \subset G$ is a closed subset and is a subgroup then: ① H is an embedded Lie subgroup.

② G/H is a manifold.

i.e. there exists a unique manifold structure on G/H s.t. the quotient map

$G \longrightarrow G/H$ is a submersion.

(moreover, it's a (right) principal H bundle)

Its manifold topology is the quotient topology.

③ Left multiplication induces a smooth action $G \circlearrowright G/H$.

2nd hour

Thm. If $G \circlearrowright M$, $m \in M$, $G_m := \mathrm{Stab}(m)$:

$G_m \subseteq G$ is a closed subgroup, hence a closed embedded sub Lie group and $G \circlearrowright G/G_m$ is a smooth action on a mfd.

Note: $G/G_m \longrightarrow G \cdot m$ is a G-equivariant bijection.
$aG_m \longmapsto a \cdot m$

Moreover: this map defines a weak embedding of G/G_m into M
i.e. $G \cdot m$ is a weakly embedded submfld of M
and $G/G_m \longrightarrow G \cdot m$ is a G-equivariant diffeomorphism.

Moreover $\mathfrak{g}_m := \mathrm{Lie}(G_m)$ is equal to

$$\{ \xi \in \mathfrak{g} \mid \xi_M|_m = 0 \}$$

and
$$T_m(G\cdot m) = \{ \xi_M|_m \mid \xi \in \mathfrak{g} \}$$

where $\xi_M \in \mathfrak{X}(M)$ is the vector field corresponding to ξ.

Back to adjoint & coadjoint actions

$G \circlearrowright G$ by
$$G \longrightarrow \mathrm{Aut}(G)$$
$$a \longmapsto (g \mapsto a g a^{-1})$$
a (Lie) group homomorphism.

Its isotropy action on $T_1 G = \mathfrak{g}$:

$\mathrm{Ad} : G \circlearrowright \mathfrak{g}$, $\quad \mathrm{Ad} : G \longrightarrow \boxed{\mathrm{Aut}(\mathfrak{g})}$ e.g. for $\mathfrak{g}$ as a vector space. (or as a Lie algebra.)

a Lie group homomorphism

(digression: $\exists!$ Lie gp str on $\mathrm{Aut}(G)$ s.t.
$\mathbb{R}^n \supset U \xrightarrow{p} \mathrm{Aut}(G)$
p is smooth iff $U \times G \to G$ is smooth. (check))

Its linearization at 1:

$\mathrm{ad} : \mathfrak{g} \circlearrowright \mathfrak{g}$, $\quad \mathrm{ad} : \mathfrak{g} \longrightarrow \boxed{\mathrm{End}(\mathfrak{g})}$ e.g. for $\mathfrak{g}$ as a vector space. (or as a Lie algebra.)

a Lie algebra homomorphism

from $\mathfrak{g}$ w/ its Lie algebra structure to $\mathrm{End}(\mathfrak{g})$ w/ the commutator.

In fact $\quad \mathrm{ad} : \xi \longmapsto (\eta \mapsto [\xi, \eta])$

The dual Lie group G-action and Lie algebra $\mathfrak{g}$-action on $\mathfrak{g}^*$: e.g. as a vector space

coadjoint action:

$\mathrm{Ad}^*: G \circlearrowright \mathfrak{g}$ $\quad\quad$ $\mathrm{Ad}^*: G \longrightarrow \mathrm{Aut}\,\mathfrak{g}^*$ a Lie group homomorphism

$$\langle \underbrace{\mathrm{Ad}^*(a)(\lambda)}_{\text{in } \mathfrak{g}^*}, \underbrace{\xi}_{\text{in } \mathfrak{g}} \rangle = \langle \lambda, \mathrm{Ad}(a^{-1})\eta \rangle$$

$\mathrm{ad}^*: \mathfrak{g} \circlearrowright \mathfrak{g}^*$ $\quad\quad$ $\mathrm{ad}^*: \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}^*)$ a Lie algebra homomorphism

$$\langle \mathrm{ad}^*(\xi)(\lambda), \eta \rangle = \langle \lambda, -[\xi, \eta] \rangle$$

Here: an action of a Lie algebra $\mathfrak{g}$ on a vector space W is a map $\xi \longmapsto (\rho(\xi): W \to W)$, $\xi \in \mathfrak{g}$

s.t. $\rho([\xi,\eta]) = \rho(\xi)\rho(\eta) - \rho(\eta)\rho(\xi)$.

i.e. it's a Lie algebra homomorphism

$$\mathfrak{g} \longrightarrow \underbrace{\mathrm{End}\, W}$$

:= linear maps $W \to W$ with $[A,B] := AB - BA$.

Given a linear action of a Lie group G on a v. space W

$$G \longrightarrow \mathrm{Aut}(W) := \left\{ \begin{matrix} \text{invertible} \\ \text{linear maps} \\ W \to W \end{matrix} \right\}$$

its linearization is a Lie algebra action on W

$$\mathfrak{g} \xrightarrow[\xi \mapsto \xi^{\#}]{} \mathrm{End}(W)$$

Note: we identify $\underset{\text{v. fields on } W}{\underset{\parallel}{\mathfrak{X}(W)}}$ with a C^∞ map $W \to W$.

w/ this identification, $\mathrm{End}(W) \subset \mathfrak{X}(W)$ are vector fields whose coefficients wrt a basis are linear functions.

Check: ~~does~~ this inclusion takes the commutator to the negative of the Lie bracket of vector fields.

Important: $G \circlearrowright W$ gives rise to an anti Lie algebra homomorphism

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{X}(W) \\ \xi &\longmapsto \xi_W \end{aligned}$$

under this identification, $\forall \xi \in \mathfrak{g}$ $\quad \xi^{\#} = \xi_W$.

at each $w \in W$

$$\underbrace{\xi^{\#}(w)}_{\text{as an element of } W} = \left.\frac{d}{dt}\right|_{t=0} (\exp(t\xi)\cdot w) = \underbrace{\left.\xi_W\right|_w}_{\text{as an element of } T_w W}$$

Let $\xi^{\#} = \xi_{\mathfrak{g}^*}$ for $\xi \in \mathfrak{g}$, be the vector fields on $\mathfrak{g}^*$ that generate the coadjoint action.

Then $\forall \lambda \in \mathfrak{g}^*$, identifying $T_\lambda \mathfrak{g}^* = \mathfrak{g}^*$, we have $\xi^{\#}\big|_\lambda = \mathrm{ad}^*(\xi)(\lambda)$

<u>Kirillov-Kostant-Souriau</u>.

Let $M \subset \mathfrak{g}^*$ be a coadjoint orbit.
Then there exists a unique sympl. form ω on M s.t. $\forall \lambda \in M$ $\forall \xi, \eta \in \mathfrak{g}$

$$\omega\big|_\lambda (\xi^{\#}, \eta^{\#}) = -\langle \underbrace{\lambda}_{\mathfrak{g}^*}, \underbrace{[\xi, \eta]}_{\mathfrak{g}} \rangle$$

The G action on M is symplectic, the inclusion $\mu : M \hookrightarrow \mathfrak{g}^*$ is a momentum map.

<u>proof</u>.
Uniqueness follows from: for each $\lambda \in M$

$$\{ \xi^{\#}\big|_\lambda \mid \xi \in \mathfrak{g} \} = T_\lambda M$$

Fix $\lambda \in M$.

$$\begin{array}{ccc} \xi, \eta & \longmapsto & -\langle \lambda, [\xi, \eta] \rangle \\ \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathbb{R} \end{array}$$

is an antisymmetric bilinear form on $\mathfrak{g}$.

We'll show: For each $\xi \in \mathfrak{g}$

$$\langle \lambda, [\xi, \eta] \rangle = 0 \;\; \forall \eta \in \mathfrak{g} \quad \overset{\text{we'll show}}{\Longleftrightarrow} \quad \xi^{\#}\big|_{\lambda} = 0$$

Recall:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, two heads, "\xi \mapsto \xi^{\#}|_{\lambda}"] \arrow[d] & & T_{\lambda} M \\ \mathfrak{g}/\mathfrak{g}_{\lambda} \arrow[urr, "\cong"] & & \end{tikzcd}$$

where $\mathfrak{g}_{\lambda} = \{ \xi \mid \xi^{\#}|_{\lambda} = 0 \}$

so: By the direction $\Leftarrow$, our formula defines a bilinear form on $T_{\lambda} M$.

By the direction $\Rightarrow$, this bilinear form is nondegenerate.

after-class discussion

take $G = GL(n) \circlearrowright W = \mathbb{R}^n$

$$G \longrightarrow \mathrm{Aut}(W)$$

$$\mathfrak{g} = \mathfrak{gl}(n) \circlearrowright W = \mathbb{R}^n$$

$$\mathfrak{g} \longrightarrow \mathrm{End}(W)$$

e.g. $\xi \in \mathbb{R}^{n \times n}$
that is not invertible.

$\exp(t\xi) \in \mathbb{R}^{n \times n}$ is invertible.

$$\underbrace{\xi \cdot w}_{\text{not invertible}} = \frac{d}{dt}\Big|_{t=0} \big(\underbrace{\exp(t\xi)}_{\text{invertible } \forall t} \cdot w \big)$$