MAT1344 29/1/2021 1-2 pm EST.

welcome! :)

please submit your 1-page processing of ~~the~~ some material if you're registered.

Reference for Lie gps & their actions: John Lee "Intro to smooth mfds", ch. 7, 8, 19, 20, 21.

Reference for compact Lie groups: Adams "Lectures on Lie groups".

orbits are weakly embedded:
- not in John Lee?
- explained in my 2011 paper w/ Iglesias
- consequence of "orbit theorem" of Sussman & Stefan mid 1970s.

Coadjoint orbits.

$Ad^* : G \circlearrowright \mathfrak{g}^*$ coadjoint action.

$\mathfrak{g}^* \supset \mathcal{O}$ coadjoint orbit.

(weakly embedded submfld).

$\xi \in \mathfrak{g} \rightsquigarrow \xi^\# \in \mathfrak{X}(\mathfrak{g}^*)$

$$\forall \lambda \in \mathfrak{g}^* \qquad \begin{array}{ccc} T_\lambda \mathfrak{g}^* & \cong & \mathfrak{g}^* \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \xi^\#\big|_\lambda & \longleftrightarrow & ad^*(\xi)(\lambda) \end{array}$$

$$\langle \underbrace{ad^*(\xi)(\lambda)}_{\mathfrak{g}^*}, \underbrace{\eta}_{\mathfrak{g}} \rangle = -\langle \lambda, [\xi, \eta] \rangle$$

For λ in the coadjoint orbit $\mathcal{O}$:

$$T_\lambda \mathcal{O} = \left\{ \xi^\#\big|_\lambda \;\middle|\; \xi \in \mathfrak{g} \right\}$$

$$B_\lambda := (\xi, \eta) \longmapsto -\langle \lambda, [\xi, \eta] \rangle$$

defines an antisymmetric bilinear map $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$.

$$\xi^\#\big|_\lambda = 0 \iff ad^*(\xi)(\lambda) = 0 \text{ in } \mathfrak{g}^*$$

$$\iff \forall \eta \in \mathfrak{g} \quad \underbrace{\langle ad^*(\xi)(\lambda), \eta \rangle}_{\substack{\shortparallel \\ -\langle \lambda, [\xi, \eta] \rangle}} = 0$$

$$\iff \forall \eta \quad B_\lambda(\xi, \eta) = 0$$

$\therefore \quad \omega(\xi^\#, \eta^\#)(\lambda) := -\langle \lambda, [\xi,\eta] \rangle$

is well defined and nondegenerate on $T_\lambda \mathcal{O}$.

Closedness of 2-forms:

For a 2-form ω on a mfld M

and vector fields X_1, X_2, X_3,

$$(d\omega)(X_1, X_2, X_3) = \sum_{\text{cyclic}} X_1\, \omega(X_2, X_3) - \sum_{\text{cyclic}} \omega\big([X_1, X_2], X_3\big)$$

Closedness of the Kirilov-Kostant-Souriau form ω on a coadjoint orbit $\mathcal{O}$.

$\forall \xi, \eta, \zeta \in \mathfrak{g}$

$$(d\omega)(\xi^\#, \eta^\#, \zeta^\#) = \underbrace{\sum_{\text{cyclic}} \xi^\# \omega(\eta^\#, \zeta^\#)}_{1\text{st}} - \underbrace{\sum_{\text{cyclic}} \omega\big([\xi^\#, \eta^\#], \zeta^\#\big)}_{2\text{nd}}$$

$$1^{\text{st}} = \sum_{\text{cyclic}} \xi^\# \big(\lambda \mapsto -\langle \lambda, [\eta, \zeta] \rangle\big)$$

$$= \sum_{\text{cyclic}} -\langle ad^*(\xi)(\lambda), [\eta, \zeta] \rangle = \sum_{\text{cyclic}} \langle \lambda, [\xi, [\eta, \zeta]] \rangle$$

$= 0$ by Jacobi

$$2^{\text{nd}} = \sum_{\text{cyclic}} \omega\big(-[\xi,\eta]^\#, \zeta^\#\big) = \langle \lambda, [[\xi,\eta], \zeta] \rangle = 0$$

by Jacobi

The momentum map. $\mu : \mathcal{O} \hookrightarrow \mathfrak{g}^*$

is equivariant.

Hamilton's equation:

$\forall \xi \in \mathfrak{g}$ $\quad \mu^\xi(\lambda) = \langle \lambda, \xi \rangle$

$$\left(d\mu^\xi \Big|_\lambda \right)(\eta^\#) = \langle ad^*(\eta)(\lambda), \xi \rangle$$
$$= \langle \lambda, -[\eta, \xi] \rangle$$
$$= -\omega(\xi^\#, \eta^\#)(\lambda)$$
$$= -\left(\iota_{\xi^\#}\omega \right)(\eta^\#)(\lambda)$$

$$d\mu^\xi \Big|_\lambda = -\iota_{\xi^\#}\omega \Big|_\lambda$$

which is Hamilton's equation.

Kostant-Souriau theorem
about transitive Hamiltonian group actions

$G \circlearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}^*$

transitive

then $\quad \mathcal{O} := \text{image } \mu$ is a coadjoint orbit. ✓

$\mu : M \to \mathcal{O}$ is a covering map,

& $\quad \mu^* \omega_{KKS} = \omega$.

The s. form on $\mathcal{O}$

proof Fix $m \in M$, let $\lambda := \mu(m) \in \mathfrak{g}^*$.

Let $G_m := \mathrm{stab}(m)$ & $G_\lambda := \mathrm{stab}(\lambda)$

$\mathcal{O} \cong G/G_\lambda$ by $g \cdot \lambda \leftrightarrow g G_\lambda$

$M \cong G/G_m$ by $g \cdot m \leftrightarrow g G_m$.

Get:

$$\begin{tikzcd} M \arrow[r, equal, "\sim"] \arrow[d, "\mu"'] & G/G_m \arrow[d] & g G_m \arrow[d] \\ \mathcal{O} \arrow[r, equal] & G/G_\lambda & g G_\lambda \end{tikzcd}$$

$$\begin{array}{c} G \\ \downarrow \\ G/G_m \end{array} \qquad \begin{array}{c} G \\ \downarrow \\ G/G_\lambda \end{array}$$

are fibre bundles.
in fact, principal bundles
wrt the right actions of G_m & G_λ.

$$\begin{array}{c} G/G_m \\ \downarrow \\ G/G_\lambda \end{array}$$

is also a fibre bundle.
(with fibre $\approx G_\lambda / G_m$)

Recall: a covering map is a fibre bundle
whose fibre is $\underbrace{\text{discrete}}_{\text{zero dimensional}}$.

To finish, it is enough to show that $\underbrace{\mu^* \omega_{KKS}}_{\text{tangents to fibres "don't have friends"}} = \underbrace{\omega}_{\text{nondegenerate: "every nonzero vector has a friend"}}$.

Indeed:

so the fibres would have to be zero dimensional

In fact, for any $G \circlearrowright (M, \omega)$

transitive $\uparrow$ closed 2-form

and (equivariant) momentum map $\mu : M \to \mathfrak{g}^*$

$$d\mu^{\xi} = -\iota(\xi_M)\,\omega$$

we'll have: image $= \mathcal{O}$ an orbit

$$\mu^* \omega_{KKS} = \omega .$$

proof of $\nearrow$: $\forall \xi, \zeta \in \mathfrak{g}$

Hamilton's eq

$$\omega(\xi_M, \eta_M)(m) = \left(\iota(\xi_M)\,\omega\right)(\eta_M)\Big|_m = -\left(d\mu^{\xi}\Big|_m\right)(\eta_M)$$

$$= -\left(\eta_M \mu^{\xi}\right)(m) = -\frac{d}{dt}\Big|_{t=0} \underline{\mu^{\xi}\left((\exp t\zeta)\cdot m\right)}$$

$$= -\frac{d}{dt}\Big|_{t=0} \left\langle \mu\left((\exp t\zeta)\cdot m\right), \xi \right\rangle$$

$$\underset{\text{equivariance}}{=} -\frac{d}{dt}\Big|_{t=0} \left\langle Ad^*(\exp t\zeta)\left(\underbrace{\mu(m)}_{\nu}\right), \xi \right\rangle$$

$$\underset{\text{def of } \mathrm{Ad}^*}{=} -\frac{d}{dt}\Big|_{t=0} \left\langle \lambda, \mathrm{Ad}\left(\exp(-t\eta)\right)(\xi) \right\rangle$$

$$= - \left\langle \lambda, \underbrace{\mathrm{ad}(-\eta)(\xi)}_{-[\eta, \xi]} \right\rangle$$

$$= - \left\langle \lambda, [\xi, \eta] \right\rangle$$

$$= \omega_{KKS}\left(\xi^{\#}, \eta^{\#}\right)$$

$$\underset{\text{equivariance}}{=} \mu^* \omega_K \left(\xi_M, \eta_M \right)$$

QED.

office hour.

$$H \subset K \subset G$$

closed subgroups.

G/H, G/K are manifolds,

$$\begin{array}{c} g \\ \downarrow \\ gH \end{array} \quad \boxed{\begin{array}{c} G \\ \downarrow \\ G/H \end{array}} \qquad \begin{array}{c} G \\ \downarrow \\ G/K \end{array}$$

are principal bundle

$$\begin{array}{c} gH \\ \downarrow \\ gK \end{array} \quad G \circlearrowright \boxed{\begin{array}{c} G/H \\ \downarrow \\ G/K \end{array}}$$

is a G-equivariant fibre bundle.

$$\begin{array}{c} G \\ \downarrow \\ G/K \end{array}$$ (right) principal K-bundle (fibres $\approx K$)

More generally:

$$\begin{array}{c} P \\ \downarrow \\ N \end{array}$$ right principal K-bundle and $K \circlearrowright F$

$\Longrightarrow$ associated bundle: $$\begin{array}{c} P \times_K F \\ \downarrow \\ N \end{array}$$ is a fibre bundle with fibre F

$P \times_K F \ni [p, f]$ $\quad$ $p \in P$, $f \in F$

$\forall k \in K \quad [pk, f] = [p, k \cdot f]$

our case:

$$\begin{array}{c} G \\ \downarrow \\ G/K \end{array} \quad \text{right princ. } K\text{-bundle.}$$

$K \circlearrowright K/H$

$\rightsquigarrow \quad G \times_K K/H$ associated bundle.

$$\begin{array}{c} G \times_K K/H \\ \wr\wr \\ G/H \end{array}$$

$[g, kH] \longleftrightarrow gkH$.

$$\begin{array}{c} G/H \\ \downarrow \\ G/K \end{array}$$ a fibre bunde with fibre K/H

$$\begin{array}{c} G/H \\ \downarrow \\ G/K \end{array} \qquad \begin{array}{c} gH \\ \downarrow \\ gK \end{array}$$

fibre over gK

$= \{ aH \mid aH = gK \}$

$= \{ aH \mid a \in gK \}$

$= gK/H \cong K/H$

one reference : appendix B of the 2002 book by Ginzburg-Guillemin-Karshon about Lie group actions.

for principal bundles, and for this being an immersion.

$G \circlearrowright M$ $\quad m \in M$, $\mathrm{Stab}(m) = G_m$

$\Rightarrow \quad G/G_m \hookrightarrow M$

<u>compact Lie group</u> G.

$T \subset G$ maximal torus.

$$Ad^* : G \circlearrowright \mathfrak{g}^*$$

$\exists$ fundamental domain $\mathfrak{t}^*_+ \subseteq \mathfrak{g}^*$

"positive Weyl chamber"

G semisimple $(\dim \underbrace{Z(G)}_{\text{the centre}} = 0)$: $\mathfrak{t}^*_+ \cong \mathbb{R}^k_{\geq 0}$

$k = \dim T$.

$\therefore$ coadjoint orbits are parametrized by $\lambda \in \mathfrak{t}^*_+$.

$\mathrm{stab}(\lambda) = T$ for all $\lambda \in \mathring{\mathfrak{t}}^*_+ \cong \mathbb{R}^k_{>0}$

We get a family

$$i_\lambda : G/T \hookrightarrow \mathfrak{g}^*$$

For $\lambda \in \mathring{t}^*_+ \approx \mathbb{R}^2_{>0}$.

$$gT \longmapsto Ad^*(g)(\lambda)$$

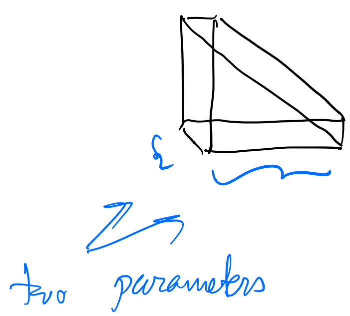
$\forall \lambda$ this is a diffeo $G/T \xrightarrow{\simeq} \mathcal{O}_\lambda$

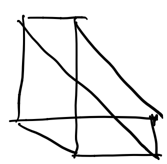
$$:= Ad^*(G)(\lambda)$$

Momentum image for the T-action:

for $G = su(3)$

$\dim G/T = 6$





$$T \subset G \circlearrowright \mathcal{O} \hookrightarrow \mathfrak{g}^*$$

$T \circlearrowright \mathcal{O}$
has isolated fixed points

Thm: $(S^1)^k \cong T \circlearrowright (M, \omega) \xrightarrow{\mu} t^*$

compact connected

isolated fixed pts $\Rightarrow$ M is simply connected.

Example for principal bundles & associated bundles.

$$\begin{array}{c} E \\ \downarrow \\ N \end{array}$$

vector bundle of $\mathbb{R}$

rank $= k$

Its frame bundle: $\begin{array}{c} P \\ \downarrow \\ N \end{array} \circlearrowleft Gl(k)$

$$P\big|_n := \{ \mathbb{R}^k \xrightarrow[\text{linear iso}]{\cong} E\big|_n \} \circlearrowleft Gl(k)$$

by precomposition

$$\begin{array}{c} P \\ \downarrow \\ N \end{array} \circlearrowleft Gl(k) \qquad Gl(k) \curvearrowright \mathbb{R}^k$$

$\rightsquigarrow$ the associated bundle:

$$[p, x] \in P \times_{Gl(k)} \mathbb{R}^k \xrightarrow{\cong} E$$

$$\downarrow$$

$$N$$

$p \in P$
$x \in \mathbb{R}^k$

$[pg, x] = [p, gx]$ for $g \in Gl(k)$.

$[p, x] \longmapsto p(x)$

$p : \mathbb{R}^k \to E\big|_m$