MAT1344 3/2/2021 12-2 pm EST.

Example. G = the Heisenberg group.

$V = \mathbb{R}^n = V^*$ via $\langle\, , \rangle$ = standard inner product

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$a = (a_1, \dots, a_n) \in V^*$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in V$

$c \in \mathbb{R}$.

$$\begin{bmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a' & c' \\ 0 & I & b' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & c+c'+\langle a, b'\rangle \\ 0 & I & b+b' \\ 0 & 0 & 1 \end{bmatrix}$$

$$1 \longrightarrow \mathbb{R} \longrightarrow G \longrightarrow (\mathbb{R}^{2n}, +) \longrightarrow 1$$

central extension

Its Lie algebra: $\mathfrak{g} = \left\{ \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \right\}$ $\alpha = (\alpha_1, \dots, \alpha_n)$ $\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$ $\gamma \in \mathbb{R}$

{\mathfrak g}

$$\exp\left(\begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & \alpha & \gamma + \frac{1}{2}\langle \alpha, \beta\rangle \\ 0 & I & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Ad}\left(\begin{bmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha & \gamma + \langle a, \beta\rangle - \langle \alpha, b\rangle \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}$$

Adjoint orbits: $\begin{cases} \{\alpha\} \times \{\beta\} \times \mathbb{R} & \text{if } \alpha, \beta \text{ not both } 0 \\ \{0\} \times \{0\} \times \{\gamma\} & \text{if } \alpha = \beta = 0 \end{cases}$

Their dimension: 1 or 0.

$\uparrow$ not symplectic!

$$\mathfrak{g}^* \cong \mathbb{R}^{2n+1} \ni (\alpha^*, \beta^*, \gamma^*) : \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \mapsto \langle \alpha^*, \alpha \rangle + \langle \beta^*, \beta \rangle + \langle \gamma^*, \gamma \rangle$$

$\alpha^*, \beta^* \in \mathbb{R}^n$, $\gamma^* \in \mathbb{R}$

The coadjoint action:

$$\Big\langle \underbrace{Ad^*\begin{pmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{pmatrix} (\alpha^*, \beta^*, \gamma^*)}_{\text{in } \mathfrak{g}^*}, \underbrace{\begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}}_{\text{in } \mathfrak{g}} \Big\rangle$$

$$= \Big\langle \underbrace{(\alpha^*, \beta^*, \gamma^*)}_{\text{in } \mathfrak{g}^*}, \underbrace{Ad\left(\begin{bmatrix} 1 & -a & -c + \langle a, b \rangle \\ 0 & I & -b \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}}_{\text{in } \mathfrak{g}} \Big\rangle$$

$$= \Big\langle (\alpha^*, \beta^*, \gamma^*), \begin{bmatrix} 0 & \alpha & \gamma - \langle a, \beta \rangle + \langle \alpha, b \rangle \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix} \Big\rangle$$

$$= \langle \alpha^*, \alpha \rangle + \langle \beta^*, \beta \rangle + \gamma^* (\gamma - \langle a, \beta \rangle + \langle \alpha, b \rangle)$$

$\mathbb{R}^n$ $\mathbb{R}^n$

$$= \langle \alpha^* + \gamma^* b, \alpha \rangle + \langle \beta^* - \gamma^* a, \beta \rangle + \gamma^* \gamma$$

Thus, $$\underbrace{Ad^*\begin{pmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{pmatrix}}_{\text{in } G} \underbrace{(\alpha^*, \beta^*, \gamma^*)}_{\text{in } \mathfrak{g}^*} = (\overbrace{\alpha^* + \gamma^* b}^{\mathbb{R}^n}, \overbrace{\beta^* - \gamma^* a}^{\mathbb{R}^n}, \overbrace{\gamma^*}^{\mathbb{R}})$$

<u>Coadjoint orbits</u>: $\mathbb{R}^{2n} \times \{\gamma^*\}$ if $\gamma^* \neq 0$ $\leftarrow$ dim $= 2n$

singletons $\{(\alpha^*, \beta^*, 0)\}$ if $\gamma^* = 0$ $\leftarrow$ dim $= 0$

when $\gamma^* \neq 0$: $\omega_{KKS} = \frac{1}{\gamma^*} \cdot \omega_{std}$ on $\mathbb{R}^{2n}$

$H := (\mathbb{R}^{2n}, +) \circlearrowright \mathbb{R}^{2n}$ by translation

does <u>not</u> have an (equivariant) momentum map

i.e. is not a Hamiltonian action.

<u>proof</u> The group is abelian so Ad^* is trivial

so $\mu : \mathbb{R}^{2n} \longrightarrow \mathfrak{h}^*$ being equivariant means being invariant.

The action is transitive so μ invariant $\Rightarrow \mu$ constant

$\Rightarrow$ the action is trivial.

Recall:

$$1 \longrightarrow \mathbb{R} \longrightarrow \underset{\text{the Heisenberg group}}{G} \xrightarrow{\pi} \underset{(\mathbb{R}^{2n}, +)}{H} \longrightarrow 1 .$$

Let $G \xrightarrow{\pi} H \circlearrowright \mathbb{R}^{2n}$

<u>nonfaithful</u> action by translations.

<u>does</u> have an equivariant momentum map.

up to $\frac{1}{}$:

$$\mu : \mathbb{R}^{2n} \longrightarrow \mathfrak{g}^* = \{(\underset{\mathbb{R}^n}{\alpha^*}, \underset{\mathbb{R}^n}{\beta^*}, \underset{\mathbb{R}}{\gamma^*})\}$$

$$(x,y) \longmapsto (\underset{\mu^1}{-y}, \underset{\mu^2}{x}, \underset{\mu^3}{-1}) .$$

is this a momentum map?

<u>assume $n=1$</u>. basis elements for $\mathfrak{g}$ act by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0$.

$$\left(\begin{array}{ccc} G & \longrightarrow & H \\ \begin{pmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{pmatrix} & \longmapsto & \begin{array}{l} \text{translations} \\ x \mapsto x+a \\ y \mapsto y+b \end{array} \end{array} \right)$$

$$\iota\left(\frac{\partial}{\partial x}\right) dx \wedge dy = dy = -d\mu^1$$

$$\iota\left(\frac{\partial}{\partial y}\right) dx \wedge dy = -dx = -d\mu^2$$

$$\iota(0)\, dx \wedge dy = 0 = -d\mu^3$$

$\therefore$ Hamilton's equation holds.

Equivariance:

$$\begin{pmatrix} 1 & a & c \\ 0 & I & b \\ 0 & 0 & 1 \end{pmatrix} : \qquad \begin{tikzcd} (x,y) \arrow[r, mapsto, "\mu"] \arrow[d, "\text{translation}"'] & (\underset{\alpha^*}{-y}, \underset{\beta^*}{x}, \underset{\gamma^*}{-1}) \arrow[d, "\text{coadjoint}"] \\ (x+a, y+b) \arrow[r, mapsto, "\mu"] & (-y-b, x+a, -1) \end{tikzcd}$$

✓

Compact Lie groups

Invariant measures on compact Lie groups:

Let G be a compact Lie group.
Then there exists a (unique) smooth probability measure on G that is invariant under left & right translations and under inversion.

Left translations are the maps $L_a : g \mapsto ag$ for $a \in G$
Right —"— $R_a : g \mapsto ga$ —"—
inversion is the map $g \mapsto g^{-1}$.

idea. trivialize $\boxed{TG \cong G \times \mathfrak{g}}$ using

$a \cdot \xi \leftrightarrow (a, \xi)$

def. M is parallelizable if $TM \cong M \times \mathbb{R}^n$

Let $n = \dim G$.
Take any n-covector at 1:

$$0 \neq v \in \Lambda^n \mathfrak{g}^*$$

Sweep it by left translation to get a left-invariant volume form.

$$vol \in \Omega^n(G), \qquad vol\big|_a = (a^{-1})^* v$$

$$L_a : G \to G$$
$$1 \mapsto a$$
$$(L_a)_* : \mathfrak{g} \mapsto T_a \mathfrak{g}$$
$$(L_a)^* : \mathfrak{g}^* \leftarrow T_a^* \mathfrak{g}$$
$$\Lambda^n \mathfrak{g}^* \leftarrow \Lambda^n T_a^* \mathfrak{g}$$

divide by $\int_G vol$ to get a new volume form with $\int_G vol = 1$

wrt the orientation determined by vol

The measure of a "nice" set A is $\int_A \mathrm{vol}$
wrt the orientation determined by vol.

Claim. This is the unique left invt volume form
up to $\pm$
with $\int_G \mathrm{vol} = 1$.

But vol & $-$vol induce the same measure
b/c we integrate each wrt its orientation.

Call this measure μ. Then for nice $f : G \to \mathbb{R}$
$\int f \, d\mu = \int f \cdot \mathrm{vol}$
wrt the orientation determined by vol.

$\boxed{\text{Claim}}$ $d\mu$ is invt under right translations & inversion.

Because left & right translations commute
$\forall a$ $R_a^* \mathrm{vol}$ is also left-invt

This $\Rightarrow$ $R_a^* \mathrm{vol} = \pm \mathrm{vol}$
define the same measure

So $d\mu$ is invt under right translations too.

Since vol is right-invariant, $(\text{inversion})^* \mathrm{vol}$ is left-invariant, so $(\text{inversion})^* \mathrm{vol} = \pm \mathrm{vol}$ & again we get the same measure.

Examples $\quad G = S^1 = \{e^{i\theta}\}$

$$\mathrm{vol} = \frac{d\theta}{2\pi}$$

$$(\text{inversion})^* \mathrm{vol} = -\mathrm{vol}.$$

$$G = \{\pm 1\} \ltimes S^1$$

$$(\varepsilon_1, z_1)\cdot(\varepsilon_2, z_2) = \begin{cases} (\varepsilon_1\varepsilon_2, z_1 z_2) & \text{if } \varepsilon_2 = 1 \\ (\varepsilon_1\varepsilon_2, z_1^{-1} z_2) & \text{if } \varepsilon_2 = -1 \end{cases}$$

$\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$

$|z_1| = |z_2| = 1$

conjugation by $(\varepsilon, 1)$ with $\varepsilon = -1$

takes (ϵ, z) to (ϵ, z^{-1})

so it flips orientation.

so vol cannot be invariant

under both left & right translation.

The measure is given by $\underbrace{|\mathrm{vol}|}_{\text{1-density}}$

on an n dim'l mfld M:

top degree diff'l forms are sections of $\Lambda^n T^* M$.

$$\begin{matrix}\hat{M} \\ \downarrow \\ M\end{matrix}$$ the orientation double covering.

$$\begin{tikzcd} \hat{M}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow \text{lifts}"] & \hat{M}_2 \arrow[d] \\ M_1 \arrow[r, "\text{diffeomorphism}"'] & M_2 \end{tikzcd}$$

$$|\Lambda^n T^* M| := \underbrace{\underbrace{\Lambda^n T^* M}_{\text{rank 1 real line bundle}} \otimes_{\mathbb{Z}_2} \hat{M}}_{\text{rank 1 real line bundle}}$$

fibrewise!

over each point of M: $\mathbb{R} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2 = \mathbb{R}$

a <u>1-density</u> of M is a section of $|\Lambda^n T^* M|$

$n = \dim M$.

over a small open set: it's represented by a pair $(\alpha, \theta) \sim (-\alpha, -\theta)$

n-form, orientation, the opposite orientation

For any compactly supported density φ, $\int_M \varphi$ is well defined.

even if M was not orientable.

(in our case $M = G$ is parallelizable hence orientable.)

$\forall$ volume form vol determines a density $|vol|$.

Averaging.

Lemma. G compact Lie group $\circlearrowright$ linearly V finite dim'l real vector space.

Then $\exists$ G-invariant inner product $\langle , \rangle$.

proof. Take *any* inner product $\langle , \rangle'$.

Note: $\forall\, a \in G$ $\quad (u,v) \mapsto \langle a\cdot u, a\cdot v\rangle'$

is also an inner product

$$\langle u, v\rangle := \int_{a\in G} \langle a\cdot u,\ a\cdot v\rangle' \underbrace{da}_{\text{an invariant probability measure}}$$

is then a G-invariant inner product on V.

Special case. G a compact Lie group

$$\mathrm{Ad} : G \circlearrowright \mathfrak{g}$$

Then $\exists\,\mathrm{Ad}$-invariant inner product $\langle , \rangle$ on $\mathfrak{g}$.

Which we use to identify $\mathfrak{g}$ with $\mathfrak{g}^*$:

$$\mathfrak{g} \xrightarrow{\ \simeq\ } \mathfrak{g}^*$$
$$\xi \longmapsto \langle \xi, \cdot \rangle$$

an Ad-invariant inner product on $\mathfrak{g}$

Claim this linear isomorphism intertwines $\mathrm{Ad} : G \circlearrowright \mathfrak{g}$ with $\mathrm{Ad}^* : G \circlearrowright \mathfrak{g}^*$.

indeed: $\forall a \in G$

$$\mathrm{Ad}(a)(\xi) \longmapsto \langle \mathrm{Ad}(a)(\xi), \bullet \rangle$$

$$= \langle \xi, \mathrm{Ad}(a^{-1}) \bullet \rangle$$

because $\langle , \rangle$ is Ad-invariant

$$= \mathrm{Ad}^*(a)\left(\langle \xi, \bullet \rangle\right)$$

$\therefore$ choice of Ad-inv't inner product on $\mathfrak{g}$ identifies adjoint orbits with coadjoint orbits.

so adjoint orbits are even dimensional.

Example $G = SU(n) = \{A \in \mathbb{C}^{n\times n} \mid \bar{A}^T A = I \ \& \det A = 1\}$

$$= \{A \in \mathbb{C}^{n\times n} \mid H(Au, Av) = H(u,v) \ \forall u, v \in \mathbb{C}^n \ \& \det A = 1\}$$

where $H(z,w) = \sum_{j=1}^{n} \bar{z}_j w_j$.

$\mathfrak{g} = \mathfrak{su}(n) = \{$traceless anti-Hermitian matrices$\}$

$$= \{A \in \mathbb{C}^{n\times n} \mid \bar{A}^T + A = 0 \ \& \operatorname{trace} A = 0\}.$$

Can take $\langle A, B \rangle = -\mathrm{Re}(\operatorname{Trace} AB)$

discussion

$$\operatorname{Tr}(AB) = \operatorname{Tr}(-\bar{A}^T B) = -\sum_j (\bar{A}^T B)_{jj}$$

$$= -\sum_{j,l} (\bar{A}^T)_{jl} B_{lj}$$

$$= -\sum_{j,l} \bar{A}_{lj} B_{lj}$$