MAT 1344 5/2/2021 1-2 pm EST

<u>practical</u>

- doodle poll on course website for lecture times] ... not yet ...
- probably will hold lectures during reading week.
  might

<u>Recall</u>. For compact Lie gps G $\exists$ Ad-inv't inner product on $\mathfrak{g}$

$$\Rightarrow \qquad \mathfrak{g} \overset{G}{\cong} \mathfrak{g}^* \qquad G\text{-equivariant linear isomorphism}$$

adjoint orbits $\longleftrightarrow$ coadjoint orbits

Coadjoint orbits for $U(n)$: Flag manifolds

$$G = U(n) = \{ A \in \mathbb{C}^{n\times n} \mid \bar{A}^T A = I \} \qquad \text{unitary matrices}$$

$$\mathfrak{g} = \mathfrak{u}(n) = \{ A \in \mathbb{C}^{n\times n} \mid \bar{A}^T + A = 0 \} \qquad \text{anti-Hermitian matrices}$$

$$\underset{iA \leftrightarrow A}{\cong} \{ A \in \mathbb{C}^{n\times n} \mid \bar{A}^T = A \} \qquad \text{Hermitian matrices}$$

A Hermitian or anti-Hermitian $\qquad A : \mathbb{C}^n \to \mathbb{C}^n$

$g \in U(n) \qquad \mathrm{Ad}(g) : A \mapsto g A g^{-1}$

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "A"] \arrow[d, "g"'] & \mathbb{C}^n \arrow[d, "g"] \\
\mathbb{C}^n \arrow[r, "\mathrm{Ad}(g)(A)"'] & \mathbb{C}^n
\end{tikzcd}$$

<u>Spectral decomposition</u> $A : \mathbb{C}^n \to \mathbb{C}^n$ is Hermitian iff

$\exists$ orthogonal decomposition $\mathbb{C}^n = E_1 \oplus \dots \oplus E_k$

$\dim E_j = m_j \qquad j = 1, \dots, k$

and real $\lambda_1 > \lambda_2 > \dots > \lambda_k$ with respect to which

A acts by $\lambda_1 I_{E_1} \oplus \dots \oplus \lambda_k I_{E_k}$.

$\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of A

$E_1, \dots, E_k$ are their eigenspaces

$m_1, \dots, m_k$ are their multiplicities.

$\forall$ such A $\forall g \in U(n)$ $\quad Ad(g)(A) = gAg^{-1} = \lambda_1 I_{gE_1} \oplus \dots \oplus \lambda_k I_{gE_k}$.

So the adjoint orbits are the sets of isospectral matrices:

$$\mathcal{H}_{\substack{m_1, \dots, m_k \\ \lambda_1, \dots, \lambda_k}} := \{ A \text{ as above} \}$$

equivariant diffeo

Let $\quad \mathcal{H}_{m_1, \dots, m_k} := \left\{ \begin{array}{l} \text{decompositions} \\ \mathbb{C}^n = E_1 \oplus \dots \oplus E_k \\ \forall j \ \dim E_j = m_j \end{array} \right\}$

Identifying $\{\text{Hermitian matrices}\} \cong \mathfrak{g}^*$

using an Ad-invariant inner product (and mult. by i) ,

we obtain for each $m_1, \dots, m_k$ a <u>family</u> of (KKS) s.forms on $\mathcal{H}_{m_1, \dots, m_k}$ parametrized by $\{\lambda_1 > \dots > \lambda_k\}$.

$U(n) \circlearrowright \mathcal{H}_{m_1, \dots, m_k}$ transitively

stabilizer of $\mathbb{C}^{m_1} \oplus \dots \oplus \mathbb{C}^{m_k}$ is $U(m_1) \times \dots \times U(m_k)$.

So $$\mathcal{H}_{m_1, \dots, m_k} \cong U(n) \big/ U(m_1) \times \dots \times U(m_k)$$

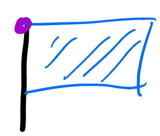
also $$\mathcal{H}_{m_1, \dots, m_k} \cong \Big\{ \underbrace{\{0\} \subseteq V_1 \subset V_2 \subset \dots \subset V_k = \mathbb{C}^n}_{\text{flags}} \Big\}$$

$$\dim V_j = m_1 + \dots + m_j$$

by $V_j = E_1 \oplus \dots \oplus E_j$, $E_j =$ orthocomplement of V_{j-1} in V_j.

Complete flags: $m_1 = \dots = m_n = 1$ $\quad k = n$

$$\{ \{0\} \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n \} \qquad \dim V_j = j \quad \forall j.$$



$k=2$, $m_1 = \ell$, $m_2 = n - \ell$:

$$\mathcal{H}_{\ell, n-\ell} \cong \mathrm{Grass}(\ell, n)$$

Grassmannian of ℓ-planes in $\mathbb{C}^n$

$\underline{\ell = 1}$: $$\mathcal{H}_{1, n-1} \cong \mathbb{CP}^{n-1}$$

complex projective space = complex lines through 0 in $\mathbb{C}^n$

For other compact Lie groups: get
"generalized flag manifolds"
"generalized Grassmannians".

$$\mathcal{H}_{m_1,\dots,m_k} \xrightarrow{\pi_j} Grass(m_j, n)$$

$j=1,\dots,k$

$$(E_1 \oplus \dots \oplus E_k) \longmapsto E_j$$

Take ω_j Kirillov-Kostant-Souriao s.form on $Grass(m_j, n)$

(e.g. normalized s.t. $\underbrace{[\omega_j]}_{\text{its deRham coh. class}}$ generates $\dfrac{H^2_{dR}(Grass(m_j, n))_{\mathbb{Z}}}{\text{image of } H^2 \text{ w/ } \mathbb{Z}\text{-coeff in } H^2_{dR}}$)

Then ω_{KKS} on $\mathcal{H}_{\substack{m_1,\dots,m_k \\ \lambda_1,\dots,\lambda_k}} \cong \mathcal{H}_{m_1,\dots,m_k}$ is

$$\lambda_1 \pi_1^* \omega_1 + \dots + \lambda_k \pi_k^* \omega_k$$

(Need to check.)

Averaging of differential forms

Lemma Let $G \circlearrowright M$.
compact Lie

Let $\alpha \in \Omega^k(M)$ be a k-form
that is exact and G-invariant.
Then α has a G-invariant primitive.
β s.t. $d\beta = \alpha$

proof. Let β' be a $(k-1)$-form s.t. $d\beta' = \alpha$.

$\forall a \in G \qquad a : M \longrightarrow M$

Consider $a^* \beta'$.

$$d\, a^*\beta' = a^* \underbrace{d\beta'}_{\alpha} = a^*\alpha = \alpha$$

α is G-invariant

Let
$$\beta := \int_{a \in G} a^*\beta' \underbrace{da}_{\text{left invariant probability measure on } G.}$$

family of $(k-1)$-forms parametrized by $a \in G$

β is a (smooth) $(k-1)$-form on M.

$$d\beta = d\int_{a\in G} a^*\beta' \, da = \int_{a\in G} \underbrace{d(a^*\beta')}_{\alpha} \, da = \alpha$$

In fact,

$$(\Omega(M)^G, d) \hookrightarrow (\Omega(M), d)$$

G-invariant diff. forms

induced

$$H^\bullet(\Omega(M)^G, d) \longrightarrow \underbrace{H^\bullet(\Omega(M), d)}_{=: H^\bullet_{dR}(M)}$$

The lemma shows that this map is an injection.

Claim. If $G \circlearrowright M$
acts trivially on cohomology
(e.g. if G is connected)

then this map is an isomorphism.

office hr. Given $m_1, \dots, m_k$ and $\lambda_1 > \dots > \lambda_k$:

$$\mathcal{H}_{m_1,\dots,m_k} := \left\{ \begin{array}{l} A : \mathbb{C}^n \to \mathbb{C}^n \\ \text{e.values } \lambda_1,\dots,\lambda_k \\ \text{multiplicities } m_1,\dots,m_k \end{array} \right\}$$

$\cong$ $\lambda_1, \dots, \lambda_k$

$$\mathcal{H}_{m_1,\dots,m_k} := \left\{ \underset{\text{decompositions}}{\mathbb{C}^n = E_1 \oplus \dots \oplus E_k} \right\}$$

Reference: Adams "Lectures on Lie groups".

$T \subset G \circlearrowright \mathfrak{g}$
maximal torus

$$\mathfrak{g}^T = \{ \eta \in \mathfrak{g} \mid Ad(a)(\eta) = a \quad \forall a \in T \}$$

$= \mathfrak{t}$

$\supseteq$ easy

$\subseteq$ requires something

$$\{ \eta \in \mathfrak{t} \mid \operatorname{stab}(\eta) = T \} \subset \mathfrak{t}$$

always $\supseteq$

open & dense

such η are called regular.

The adjoint orbit through a regular $\eta \cong G/T$

even dimension.

$T \circlearrowright \mathfrak{g} = \mathfrak{t} \oplus \underbrace{\mathfrak{m}}_{}$ complementary T-invt subspace.

$\mathfrak{t} = \mathfrak{g}^T \Rightarrow T \circlearrowright \mathfrak{m}$ has no nontrivial fixed subspace

$\Rightarrow \mathfrak{m} = \underset{T}{\mathbb{R}^2} \oplus \dots \oplus \underset{T}{\mathbb{R}^2}$

$\Rightarrow \dim \mathfrak{m}$ is even

But $\mathfrak{m} = \mathfrak{g}/\mathfrak{t} = T_1(G/T)$

another argument why adjoint orbits have even dimension when G is compact

$G = O(4) = \{ A \in \mathbb{R}^{4\times 4} \mid A^T A = I \}$.

$\mathfrak{g} = \{ A \in \mathbb{R}^{4\times 4} \mid A^T + A = 0 \}$

anti symmetric.

$$\mathfrak{t} = \left[\begin{array}{cc|cc} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & \lambda_2 \\ \multicolumn{2}{c|}{0} & -\lambda_2 & 0 \end{array}\right]$$

two such matrices are conjugate in $O(4)$
iff they differ by $\lambda_1 \longleftrightarrow -\lambda_1$

$\lambda_2 \longleftrightarrow -\lambda_2$

$\lambda_1 \longleftrightarrow \lambda_2$.

assume $\lambda_1 > \lambda_2 > 0$ $\quad \mathbb{R}^4 = E_1 \oplus E_2$

A acts on E_1 by $\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}$

on E_2 by $\begin{pmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{pmatrix}$

So (I think) the corresponding gen. flag md

$= \left\{ \begin{array}{l} \text{decompositions } \mathbb{R}^4 = E_1 \oplus E_2 \\ + \text{ orientation on each } E_1, E_2 \end{array} \right\}$

other types: $\lambda_1 = \lambda_2 > 0$, $\quad \lambda_1 > \lambda_2 = 0$, $\quad \lambda_1 = \lambda_2 = 0$.

$\mathbb{R}^k_{\geq 0} \cong \mathfrak{t}^*_+ \subset \mathfrak{g}^*$ positive Weyl chamber

fundamental domain for the coadjoint action.

The manifold structure.

$$\mathcal{H}_{m_1, \dots, m_k} = \{ E_1 \oplus \dots \oplus E_k \} \cong U(n) / U(m_1) \times \dots \times U(m_k)$$

induces a md str on $\mathcal{H}_{m_1, \dots, m_k}$

Theorem:

$$\forall \lambda_1, \dots, \lambda_k \quad \mathcal{H}_{m_1, \dots, m_k} \cong \mathcal{H}^{\lambda_1, \dots, \lambda_k}_{m_1, \dots, m_k} \hookrightarrow \mathcal{U}(n)$$

is an embedding