10/2/2021.

Welcome back!

practical.

- doodle poll: postponing
- lectures @ reading week: will announce.

HW02: please send me ≈ 1 page of math! (max 2)

→ use LaTeX (see samplefile on course website)
→ include some title, your name,
→ use your voice.

within 1 week.

Finish from last time:

$\underset{\text{compact}}{G} \circlearrowright M$

$$(\Omega^\bullet(M)^G, d) \hookrightarrow (\Omega(M), d)$$

induces a map in cohomology.

Shown: this map in cohomology is one-to-one.
(every invariant exact form has an invariant primitive)

Claim: Assume G acts trivially on $H^\bullet_{dR}(M)$. (*)
Then the map in cohomology is also onto.

(*) e.g. this holds if $G \subset \underset{\text{connected}}{\tilde{G}} \circlearrowright M$

by homotopy invariance of deRham cohomology: $(a_t)_{t\in[0,1]}$ in $\tilde{G}$, $a_0 = 1$, $a_1 = a \in G$ then all maps a_t^* on coh. are the same hence trivial.

i.e. every closed form
is cohomologous to a closed invariant form.

sketch of proof. Let α' be a closed k-form on M.

For each $g \in G$ $\quad g : M \longrightarrow M$

$$[g^* \alpha'] = g^*[\alpha'] = [\alpha']$$

by assumption

Then $$\alpha := \int_{g \in G} g^* \alpha' \; \underbrace{dg}$$

invariant prob. measure on G

$$d\alpha = d\int_{g \in G} g^*\alpha' \, dg = \int_{g \in G} \overbrace{d(g^*\alpha')}^{=0} \, dg = 0.$$

exterior derivative

$$[\alpha] = \left[\int_{g \in G} g^*\alpha' \, dg\right] = \int_{g \in G} \underbrace{[g^*\alpha']}_{=[\alpha']} \, dg = [\alpha'].$$

e.g. by viewing $\int_{g \in G}$ as a limit of sums....

Some point set topology.

Recall

- closed subset of cpct set is cpct.
- Y Hausdorff, then $\forall$ compact subset of Y is closed.

Def a Hausdorff space Y is compactly generated if $\forall$ subset C if $\forall$ compact K $K \cap C$ is closed, then C is closed.

Exercise. if a Hausdorff space Y is
metrizable
OR
locally compact
then Y is compactly generated.

for each point p of Y, each neighbourhood of p contains a compact neighbourhood of p.

A "neighbourhood" of a subset A of Y is a subset of Y that contains an open set that contains Y.

$\boxed{\text{Proper maps}}$

Let $\Psi : X \to Y$ be a continuous map of topological spaces.

Def Ψ is **proper** if $\forall$ compact subset K of Y, $\Psi^{-1}(K)$ is compact.

Note. Id : open ball in $\mathbb{R}^n$ $\to$ open ball in $\mathbb{R}^n$ is proper.

inclusion: open ball in $\mathbb{R}^n$ $\hookrightarrow \mathbb{R}^n$ is not proper.

Exercise.

- a composition of proper maps is proper
- Restriction of a proper map to a closed subset is proper.
- $\Psi : X \to Y$, $A \subset Y$, then $\Psi|_{\Psi^{-1}(A)} : \Psi^{-1}(A) \to A$ is proper
- $\Psi : X \to Y$ proper $\Rightarrow$ level sets are compact
- $\Psi : X \to Y$ continuous, closed, compact level sets $\Rightarrow$ proper.
- $\Psi : X \to Y$ continuous & proper, Y is compactly generated. Then Ψ is a closed map.
- $\Psi : X \to Y$ continuous, X compact, Y Hausdorff $\Rightarrow \Psi$ proper.

Lemma. $\underset{\text{compact}}{G} \circlearrowright M \underset{\text{open}}{\supset} U$

Then $\bigcap_{a \in G} a \cdot U$ is G-invariant & open.

proof Its complement is
$$G \cdot (M \setminus U)$$
So its enough to show that if C is closed Then $G \cdot C$ is closed.

$$\varphi : G \times M \longrightarrow M$$
$$(a, m) \longmapsto a \cdot m$$

is proper. indeed, $\underset{\text{compact}}{K \subset M}$ Then
by continuity of φ
$\varphi^{-1}(K)$ is a closed subset of $G \times (G \cdot K)$.

$G \cdot K = \varphi(G \times K)$ is compact.

continuous compact

So $\varphi^{-1}(K)$ is a closed subset of the compact set $G \times (G \cdot K)$ so it's compact.

$G \cdot C = \varphi(\underbrace{G \times C}_{\text{closed in } G \times M})$ is closed because φ is proper, hence a closed map.

proper $\rightarrow$

Digression.

for a not-necessarily compact G,

an action $G \circlearrowright M$ is <u>proper</u> if

the map $G \times M \longrightarrow M \times M$ is proper.

$(g, m) \longmapsto (m, g \cdot m)$

if G is compact, this automatically holds.

★ lots of properties of compact group actions are also true for <u>proper</u> group actions but sometimes the proofs are harder.

example $G = SE(3) \circlearrowright \mathbb{R}^3$ is proper.

$\quad\quad\quad\quad \| $

$\quad\quad \mathbb{R}^3 \rtimes SO(3)$

Smooth local linearization.

$H \circlearrowright M$, $x \in M^H$

compact

i.e. $H \cdot x = \{x\}$

$H \circlearrowright T_x M$ isotropy

Then there exist

$$\begin{pmatrix} H\text{-invariant} \\ \text{nbhd of } x \\ \text{in } M \end{pmatrix} \xrightarrow[x \mapsto 0]{H\text{-equivariant diffeo}} \begin{pmatrix} H\text{-invt nbhd} \\ \text{of } 0 \text{ in } T_x M \end{pmatrix}$$

(Moreover, if M is a complex mfld, we obtain an H-equivariant biholomorphism)

proof. a choice of a centred chart determines a diffeomorphism

$$f : \begin{pmatrix} \text{a nbhd } U \\ \text{of } x \text{ in } M \end{pmatrix} \longrightarrow \begin{pmatrix} \text{a nbhd } \Omega \\ \text{of } 0 \text{ in } T_x M \end{pmatrix}$$

s.t. $x \longmapsto 0$

and $df\big|_x = Id_{T_x M}$

Let $g := \displaystyle\int_{a \in H} a \circ f \circ a^{-1} \, da$ invariant probability measure

Here $\quad \underbrace{a}_{\text{linear isotropy action of } a\in H \text{ on } T_xM} \circ f \circ \overbrace{a^{-1}}^{\text{given action of } a\in H \text{ on } M} \; : \; a\cdot U \xrightarrow[\text{diffeo}]{} a\cdot\Omega$

So g is defined on $\underbrace{\bigcap_{a\in H} a\cdot U}_{\text{open}} \ni x$

which is an open nbhd of x in M.

$$dg\Big|_x = \underset{\text{differential}}{d}\Big(\int_{a\in H} a\circ f\circ a^{-1} \underset{\text{measure}}{da}\Big)\Big|_x$$

$$= \int_{a\in H} d\,(a\circ f\circ a^{-1})\Big|_x \, da$$

$$= \int_{a\in H} a\circ \underbrace{df\Big|_x}_{Id}\circ a^{-1} \, da.$$

$$= Id_{T_xM}.$$

We get:

$$g : \begin{pmatrix}H\text{-inv't nbhd}\\ \text{of } x \text{ in } M\end{pmatrix} \longrightarrow T_xM$$

s.t. $dg\Big|_x = Id_{T_xM}$ and $\forall\, h\in H \quad hgh^{-1} = g$

i.e. g is H-equivariant

By the inverse function theorem
$\exists$ nbhd V of x on which g is a diffeomorphism with its image.

Then $g\Big|_{\bigcap_{a \in H} a \cdot V} : \begin{pmatrix} H\text{-inv}^{\text{t}} \text{ open nbhd} \\ \text{of } x \text{ in } M \end{pmatrix} \longrightarrow T_x M$

is an H-equivariant diffeo w/ an H-inv$^{\text{t}}$ open subset $T_x M$.

✓

Corollary. Let a compact Lie group H act on a mfld M.

Let $M^H := \{H\text{- fixed points}\}$. Then

- Each connected component of M^H is a submanifold of M.
- If M is oriented & $H \circlearrowright M$ preserves orientation (e.g. this holds if $\underset{\text{connected}}{H} \subset G \circlearrowright \underset{\text{oriented}}{M}$)

  Then M^H has codimension ≥ 2 in M.
- Suppose that M/H is connected. (e.g. if M is connected).

  and suppose that $\exists$ open subset U of M on which H acts trivially.

  Then $H \circlearrowright M$ is trivial.

  proof. By assumption, M^H contains some open set.

So $\exists$ a component X of M^H that contains some open set.

BUT each component of M^H is a closed submfld of M.

by continuity of the action

consequence of local linearisation.

Because X is a submfld & contains an open set, X is an open subset of M.

Because X is open & closed & connected,

So X is a connected component of M on which H acts trivially.

$$M = X \sqcup (M \setminus X)$$

closed & open in M

$$M/H = X/H \sqcup (M \setminus X)/H$$

trivial quotient

open & closed in M/H by def of the quotient topology.

X/H is nonempty, M/H is connected

$\therefore \quad M = X$

Corollary Let $G \circlearrowright M$, G compact.

Suppose that the action is faithful and M is connected.

Then the action is locally faithful.

i.e. $\forall$ G-inv't open subset U of M, $G \circlearrowright U$ is faithful.

proof. Fix a G-inv't open subset U of M.
Let K be the kernel of the G-action on U.
Then M^K has nonempty interior
So K acts trivially on M.
Because $G \circlearrowright M$ is faithful, $K = \{1\}$.

Non-examples for noncompact group actions

$C \subsetneq \mathbb{R}$ closed subset e.g. Cantor set.
Let $f: \mathbb{R} \to \mathbb{R}$ be a bounded smooth function
w/ $C = f^{-1}(0)$.

Then $f(x) \frac{\partial}{\partial x}$ generates an $\mathbb{R}$-action on $\mathbb{R}$
w/ fixed point set $= C$
generally not a submfld.

Let $g: \mathbb{R} \to \mathbb{R}$ be $g(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$.

Then $\frac{g(x)}{1+g(x)} \frac{\partial}{\partial x}$ generates an $\mathbb{R}$-action on $\mathbb{R}$
That is faithful but not locally faithful.