12/2/2021

Hello! ☺

Reminders:

- might hold lectures next week at usual times.
- HW 2: 1-2 page of math! details last time.
  aim for doing it by Wednesday.

References for proper gp actions:
→ John Lee "Intro to smooth mflds"
→ Appendix B to Ginzburg-Guillemin-Karshon.

Recall. $G \circlearrowright M$ proper means:

$$\begin{aligned} G \times M &\longrightarrow M \times M \quad \text{is a proper map} \\ (a, m) &\longmapsto (a \cdot m, m) \end{aligned}$$

If G is compact, $\begin{aligned} G \times M &\longrightarrow M \\ (a,m) &\longmapsto a \cdot m \end{aligned}$ is proper.

hence $G \circlearrowright M$ is proper.

Exercise: $G \circlearrowright M$ proper
Then orbits are closed and M/G is Hausdorff.
(see GGK appendix B)

Principal bundles.

G, H Lie group.

a (right) principal H-bundle over a mfld M is

- a mfld P
- a map $\pi : P \to M$
- a right H-action on P s.t.

→ the H-orbits are the π-fibres

→ for each point in M $\exists$ open nbhd U & diffeo

$$\begin{tikzcd} P|_U := \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times H \arrow[dl, "\text{projection}"] \\ & U & \end{tikzcd}$$

s.t.

the diagram commutes

and the diffeo intertwines the given H-action on $\pi^{-1}(U)$

with the right-multiplication H-action on $U \times H$.

A (left) G-equivariant principal H-bundle is

$$\begin{array}{c} P \circlearrowleft H \\ \pi \downarrow \\ M \end{array}$$ as above

and a (left) G-action on P

that commutes w/ the principal H-action

(hence descends to a G-action on M).

Note If

$$\begin{array}{c} P \circlearrowleft H \\ \pi \downarrow \\ M \end{array}$$

is a principal H-bundle

and $\sigma : M \longrightarrow P$ is a smooth section

i.e. $\pi \circ \sigma = Id_M$

Then

$$\begin{tikzcd} M \times H \arrow[rr] \arrow[dr, "\text{projection}"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

$$(m, a) \longmapsto \sigma(m) \cdot a$$

is an H-equivariant diffeomorphism & the diagram commutes.

(The main point is to show that the inverse is smooth)

Example Let

$$\begin{array}{c} P \circlearrowleft H \\ \pi \downarrow \\ M \end{array}$$

be a principal H-bundle

and let Z be the centre of H.

Let Z act on P by the restriction of the H-action

Because Z is abelian, this is also a left action,

and

$$\begin{array}{c} Z \circlearrowright P \circlearrowleft H \\ \downarrow \\ M \end{array}$$

is a Z-equivariant principal H-bundle.

Warning the left H-action on P given by

$$a : p \longmapsto p \cdot a^{-1}$$

does <u>not</u> commute w/ the principal H-action if H was not abelian.

Example. Let G be a Lie group and H a closed subgroup. Then $\exists!$ mfld str on G/H s.t. $G \longrightarrow G/H$ is a submersion.

Moreover,

$$\begin{array}{c} G \circlearrowright G \circlearrowleft H \\ \downarrow \\ G/H \end{array}$$

is a G-equivariant principal H-bundle.

proof.

exercise: $G \circlearrowleft H$ is a proper action.

So G/H is Hausdorff.

Also, because G is 2nd countable, G/H is also 2nd countable.

we claim that each coset in G/H has an right H-invt open nbhd in G that is H-equivariantly diffeomorphic to $U \times H$ where $U \underset{\text{open}}{\subset} \mathbb{R}^k$ (for k = codim of H in G)

This implies that G/H $\boxed{\text{is}}$ a mfld and $G \longrightarrow G/H$ $\boxed{\text{is}}$ a principal H-bundle.

proof of the claim: decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as vector spaces.

Define

$$\begin{aligned} \varphi : \mathfrak{m} \times H &\longrightarrow G \\ (y, a) &\longmapsto \exp(y) \cdot a \end{aligned}$$

is smooth & right H-equivariant

Note. $d\Psi\big|_{(0,1)} : \mathfrak{m} \oplus \mathfrak{h} \longrightarrow \mathfrak{g}$

is the identity map.

H-equivariance & Implicit fctn thm

$\Rightarrow$ $d\Psi$ is a linear iso at each pt of $\{0\} \times H$

and Ψ restricts to a local diffeo (in particular, locally one to one)

$$\Psi\big|_{D \times H} : D \times H \longrightarrow G$$

for some nbhd D of 0 in $\mathfrak{m}$.

Claim. after possibly shrinking D, Ψ is one-to-one hence a diffeo as required.

Indeed:

otherwise $\exists$ $(u_n, a_n) \neq (v_n, b_n) \in \mathfrak{m} \times H$ s.t.

$u_n, v_n \longrightarrow 0$ and $\exp(u_n) \cdot a_n = \exp(v_n) \cdot b_n$.

Let $c_n = a_n b_n^{-1}$. Then $\exp(u_n) \cdot c_n = \exp(v_n)$.

$$c_n = \left(\exp(u_n)\right)^{-1} \cdot \exp(v_n) \longrightarrow 1$$

b/c $u_n, v_n \longrightarrow 0$

$$u_n \longrightarrow 0$$

So $(u_n, c_n) \neq (v_n, 1)$, $\Psi(u_n, c_n) = \Psi(v_n, 1)$

$(u_n, c_n) \longrightarrow (0, 1)$

$(v_n, 1) \longrightarrow (0, 1)$.

contradicting the fact that Ψ is a local diffeo near $(0,1)$.

we found $D \underset{\text{open}}{\subset} \mathfrak{m} \cong \mathbb{R}^k$ and a right H-eq. diffeo

$$D \times H \longrightarrow \begin{array}{c}\text{open} \\ \text{nbhd of } H \\ \text{in } G.\end{array}$$

left translation by $a \in G$ gives a right H-eq. diffeo

$$D \times H \longrightarrow \begin{array}{c}\text{open nbhd} \\ \text{of } g \cdot H \\ \text{in } G\end{array}$$

Vector bundles

a rank k v. bundle over a md M is

- a md E
- a map $\pi : E \to M$
- a str. of a v. space on each fibre of E

s.t. for each pt in M $\exists$ open nd U and $\exists$ diffeo

$$\begin{tikzcd} E\big|_U := \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\text{projection}"] \\ & U & \end{tikzcd}$$

s.t. the diagram commutes and the diffeo restricts to a linear isomorphism on each fibre.

associated bundle.

Given a G-equivariant principal right H-bundle

$$\begin{tikzcd} G \circlearrowright P \circlearrowleft H \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

and a linear H-repr. $H \circlearrowright W \cong \mathbb{R}^n$

The corresponding associated vector bundle is

$$E = P \times_H W := P \times W / H$$

where $a \in H$ acts on $P \times W$ by $(p, w) \mapsto (p \cdot a^{-1}, a \cdot w)$

with the map

$$\pi_E : E \longrightarrow M$$
$$[p, w] \longmapsto \pi(p)$$

w/ ~~a~~ the fibrewise vector space structure induced from W.

Each local trivialization of P, $\pi^{-1}(U) \cong U \times H$

gives a local trivialization of E:

$$\pi_E^{-1}(U) \cong (U \times H) \times_H W \cong U \times W$$

Elaboration & fixing of statement from last time

(each component of)

if $H \circlearrowright M$ compact Then $M^H \subset M$ is a submfd.

If M is oriented & H preserved orientation Then components of M^H cannot have codimension 1.

Non Example. $\mathbb{Z}_2 \circlearrowright \mathbb{R}$

$x \mapsto -x$.

The fixed point set has codim = 1 (not ≥ 2)

proof: by local linearization, assume

$\underset{\text{linear}}{H \circlearrowright V}$, V oriented, H preserves orientation.

$\underbrace{V^H}_{\text{set of } H\text{-fixed vectors}} \subset V$ is a subvector space. hence a submfld.

and $\underbrace{H \circlearrowright V/V^H}_{\text{w/out nontrivial fixed subspace.}}$ preserves orientation

If $\operatorname{codim} V^H = 1$ then $\underbrace{H}_{\text{compact}} \circlearrowright \mathbb{R}$ & preserves orientation.

H compact $\Rightarrow$ must act by ± 1.

$\Rightarrow$ cannot preserve orientation

Since each component of M^H that has codim $=0$ is a component of M,

if M is connected and $H \curvearrowright M$ is nontrivial & preserves orientation,

then each component of M^H has codim ≥ 2.

Let M be a mfld & let N be a set & let $\pi : M \to N$ be a surjective map.

Suppose that $\exists$ a mfld str on N s.t. π is a submersion

Then such a structure is unique.

Indeed, for each map $f : N \to \mathbb{R}$

f is smooth $\Leftrightarrow$ $f \circ \pi : M \to \mathbb{R}$ is smooth.

$\Rightarrow$: by smoothess of π

$\Leftarrow$: b/c π is onto & by the "submersion thm".