a-synchronous lecture

proper group actions

a group action $G \circlearrowright M$ is proper if the map
$$G \times M \longrightarrow M \times M \quad \text{is proper.}$$
$$(a, m) \longmapsto (a \cdot m, m)$$

Lemma.

① $G \circlearrowright M$, G cpct $\Rightarrow$ $G \circlearrowright M$ proper

② G not compact, M compact $\Rightarrow$ $G \circlearrowright M$ not proper

③ $G \circlearrowright M$ proper

- $H \subset G$ closed subgp, then the restricted action $H \circlearrowright M$ is proper
- $N \subset M$ ~~closed~~ invariant submfld, then the restricted action $G \circlearrowright N$ is proper

④ $G \circlearrowright M$ proper then $\forall x \in M$
$$G \longrightarrow M \quad \text{is proper}$$
$$a \longmapsto a \cdot x$$

Corollary $G \cdot x$ is closed.

⑤ $G \circlearrowright M$ is proper iff $\forall K_1, K_2 \subset M$ compact
$$\{a \in G \mid aK_1 \cap K_2 \neq \phi\} \quad \text{is compact.}$$

⑥ $G \circlearrowright M$ proper $\Rightarrow$ $\forall x \in M$ $stab(x) \subset G$ is compact.

Example. $G \circlearrowright M := G$ by left translation
$a : b \longmapsto ab$ is proper

Example $(\mathbb{C}^\times)^n \circlearrowright \mathbb{C}^n$ is not proper.
acting by coordinatewise multiplication

proposition. $G \circlearrowright M$ proper, then
- orbits are closed
- M/G is Hausdorff

Local linearization.

$H \circlearrowright M$ compact $\qquad x \in M^H := \left\{ \begin{array}{c} \text{points} \\ \text{fixed by } H \end{array} \right\}$

Then $\exists$ H-equivariant diffeomorphism

$$\left(\begin{array}{c} H\text{-inv}^{\text{t}}\text{l open nbhd} \\ \text{of } x \text{ in } M \end{array} \right) \longrightarrow \left(\begin{array}{c} H\text{-inv}^{\text{t}}\text{l open nbhd} \\ \text{of } 0 \text{ in } T_x M \end{array} \right)$$

$$x \longmapsto 0$$

Corollaries.
- M^H is a locally finite disjoint union of closed submflds of M

- Let $\underset{\cancel{\text{compact}}}{G} \circlearrowright M$ be a proper faithful action & assume M/G is connected.

Let $U \subset M$ be a G-inv't open subset.

Then $G \circlearrowright U$ is faithful.

Proof. Let K be the kernel of the G-action on U. Then K is compact.

also $K \triangleleft G$ is a normal subgroup.

Consider $K \circlearrowright M$. Then M^K is a locally finite disjoint union of closed submflds of M. & has nonempty interior.

So $\mathcal{O} :=$ interior(M^K) is nonempty and is a union of connected components of M & is both closed and open in M.

Because $K \triangleleft G$ is a normal subgroup, $\mathcal{O}$ is G-invariant.

so $\mathcal{O}/G$ is nonempty & closed & open in M/G

But M/G is connected. so $\mathcal{O}/G = M/G$

so $\mathcal{O} = M$

So K acts trivially on M.

But $G \circlearrowright M$ is faithful. So $K = \{1\}$.

Koszul's slice theorem.

$G \circlearrowright M$ proper action; $x \in M$

$H := \mathrm{Stab}(x)$.

$\llcorner$ compact.

$G \circlearrowright TM|_{G\cdot x} / T(G\cdot x) =: \nu$ the normal bundle to $G\cdot x$ in M.

restricts to $\underbrace{H \circlearrowright \nu_x}$ $= T_x M / T_x(G\cdot x)$

=: the slice representation at x

$G \circlearrowright G \circlearrowleft H$ G-equivariant principal H-bundle.

$\downarrow$

G/H

associated bundle: $G \times_H \nu_x \xrightarrow{\cong} \nu$

$[a, w] \longmapsto a\cdot w$

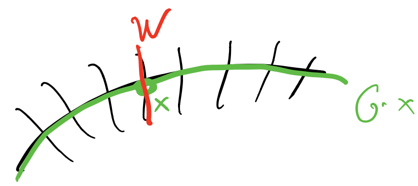
Theorem.

$\exists$ a G-equivariant diffeomorphism
from a G-inv't open nbhd of the 0-section in $G \times_H \nu_x$
to —— " —— of the orbit $G \cdot x$ in M.
that takes $[a, 0]$ to $a \cdot x$
and whose differential along the zero section
descends to the above isomorphism $G \times_H \nu_x \xrightarrow{\cong} \nu$

(using: the normal bundle to the 0-section in a $\underbrace{\text{vector bundle}}_{\text{e.g. } G\times_H V_x}$ can be identified with the bundle itself)

Proof of Koszul's slice theorem.

Decompose $T_xM = T_x(G\cdot x) \oplus \underbrace{W}_{H\text{-invariant}}$



(e.g. choose an H-invariant inner product on T_xM & take $W = \left(T_x(G\cdot x)\right)^{\perp}$)

Note:

$$\begin{tikzcd} W \arrow[r, "\text{inclusion}"'] \arrow[rr, bend right, "\cong"'] & T_xM \arrow[r, "\text{quotient}"'] & V_x \end{tikzcd}$$

By local linearization $\exists$ H-equivariant diffeomorphism

$$F : \begin{pmatrix} H\text{-invt} \\ \text{open nbhd} \\ \text{of } 0 \text{ in } T_xM \end{pmatrix} \longrightarrow \begin{pmatrix} H\text{-invt} \\ \text{open nbhd} \\ \text{of } x \text{ in } M \end{pmatrix}$$

$$0 \longmapsto x$$

Fix an H-invt inner product W

Let $D \subset W$ be an open ball about 0 contained in the domain of F.

The "Sweeping" map $\Psi : G \times_H D \longrightarrow M$ is.....

$$[a, v] \longmapsto a \cdot F(v)$$

(Reminder: $G \times_H D = \dfrac{G \times D}{H}$ where $h \in H$ acts on $G \times D$ by $(a, v) \mapsto (ah^{-1}, h \cdot v)$

It's a disc-bundle over G/H.)

Ψ is well defined,

G-equivariant,

is a local diffeomorphism near $[1, 0]$

(at $[1,0]$ we can identify $d\Psi|_{[1,0]}$ with $dF|_0 : T_x M \to T_x M$ which is the identity map)

After shrinking D, WLG Ψ is a local diffeo near each $[1, v]$ for $v \in D$.

By G-equivariance, Ψ is a local diffeo on all of $G \times_H D$.

We claim that, after further shinking D, Ψ is one to one, hence a G-equivariant diffeo of $G \times_H D$ w/ a G-invariant open subset of M that contains $G \cdot x$, as required.

proof of the claim:

otherwise $\exists\ u_n, v_n \longrightarrow 0$ in W

and $\exists\ a_n, b_n \in G$ s.t.

s.t. $[a_n, u_n] \neq [b_n, v_n]$ in $G \times_H W$

and $a_n \cdot F(u_n) = b_n \cdot F(v_n)$

Let $c_n := b_n^{-1} a_n$. Then $[c_n, u_n] \neq [1, v_n]$

and $c_n \cdot F(u_n) = F(v_n)$

$v_n \longrightarrow 0$. So $F(v_n) \longrightarrow x$.

so $\{F(v_n)\}$ is contained in a compact subset of M, namely, in $\{F(v_n)\}_{n\in\mathbb{N}} \cup \{x\}$.

Under the proper map $G \times M \longrightarrow M \times M$

$(a, m) \longmapsto (a \cdot m, m)$

$$(c_n, F(u_n)) \longmapsto \underbrace{(\underbrace{c_n \cdot F(u_n)}_{F(v_n)}, F(u_n))} \xrightarrow[\text{as } n\to\infty]{} (F(0), F(0)) = (x, x)$$

because $u_n, v_n \longrightarrow 0$

So $\{(c_n, F(u_n))\}_{n\in\mathbb{N}}$ is contained in the preimage of a converging sequence in $M \times M$,

hence it's contained in the preimage of a compact subset of $M \times M$;

because the map $G \times M \to M \times M$ is proper,

$\{(c_n, F(u_n))\}_{n \in \mathbb{N}}$ is contained in a compact subset of $G \times M$.

So after passing to a subsequence we may assume that $c_n \xrightarrow[n \to \infty]{} c_\infty$.

(converges in G)

$$c_\infty \cdot x = \lim_{n \to \infty} \underbrace{c_n \cdot F(u_n)}_{F(v_n)} = \lim_{n \to \infty} F(v_n) = F(0) = x$$

So $c_\infty \in \mathrm{stab}(x) = H$.

We now have: $[c_n, u_n] \neq [1, v_n]$ in $G \times_H W$

but $c_n \cdot F(u_n) = F(v_n)$ i.e. $\Psi([c_n, u_n]) = \Psi([1, v_n])$

as $n \to \infty$,

$$\begin{cases} [c_n, u_n] \longrightarrow [c_\infty, 0] = [1, 0] \quad (c_\infty \in H) \\ \text{and} \\ [1, v_n] \longrightarrow [1, 0] \end{cases}$$

This contradicts the fact that Ψ is one-to-one near $[1, 0]$

∎

We proved Koszul's slice theorem.

In particular:

$G \circlearrowright M$ proper, $x \in M$

$H = \mathrm{stab}\, x$

Then $\exists$ a G-inv't open nbhd of $G \cdot x$ in M

$\underset{\substack{\text{G-equivariantly} \\ \text{diffeomorphic}}}{\cong}$ $G \times_H D$

for $\underset{\text{linear}}{H \circlearrowright W} \underset{\text{disc}}{\supset D}$

with $W \cong \nu_x = T_x M / T_x(G \cdot x)$.

Corollary of Koszul's slice thm.

$M_{princ} := \{$point in M whose slice representation is trivial$\}$

$H \underset{\text{closed}}{\subset} G$. $M_{(H)} := \{ x \in M \mid \mathrm{stab}(x)$ is conjugate to $H \}$

<u>Note</u>. $M_{(H)} \subset M$ is G-invariant.

Lemma ① $M_{\text{princ}} \subset M$

is a G-invariant open dense set.

② If M/G is connected, then

$\exists$ closed subgroup $H \subset G$ s.t.

$M_{\text{princ}} = M_{(H)}$,

and M_{princ}/G is connected.

Def. The connected components of $M_{(H)}$

for closed subgps H of G

are called the orbit-type strata in M.

If M and G are connected

M_{princ} is the principal stratum in M.

Sometimes people work w/ $M_{(H)}$ themselves

or w/ unions of similar components...

Lemma ① $M_{princ} \subset M$ is a G-invariant open dense set.

② If M/G is connected, then $\exists$ closed subgroup $H \subset G$ s.t. $M_{princ} = M_{(H)}$, and M_{princ}/G is connected.

Main ingredients in ~~steps of~~ the proof of the lemma

- a subgroup of H that is conjugate to H must be equal to H.

Here $H \subset G$ compact subgroup

- For x with $Stab(x) = H$, $x \in M_{princ}$ iff $x \in \text{interior}(M_{(H)})$

- For each orbit $G \cdot x$ in M there exists a closed subgroup $H \subset G$ and a G-invariant open nbhd U of $G \cdot x$ in M s.t. $U \cap M_{(H)}$ is open & dense in U and $(U \cap M_{(H)})/G$ is connected.

Use Koszul's slice theorem.

The claim for $G \times_H W$ follows from the similar claim for the unit sphere in W wrt any H-invariant inner product.