26/2/2021, 1-2 pm EST.

Cotangent bundles.

N manifold. $\dim = n$

$$T^*N = \bigsqcup_{x \in N} \underbrace{T_x^* N}_{(T_x N)^*}$$

write $\varphi_x \in T_x^* N$ for $x \in N$

$\pi : T^*N \longrightarrow N$ $\qquad \varphi_x \longmapsto x$

Local coordinates $U \subset N$: $(q_1, \cdots, q_n)$

Adapted coordinates on $T^*N|_U := \pi^{-1}(U) = T^*U$ $\qquad (q_1, \cdots, q_n, p_1, \cdots, p_n)$

$\forall x \in U \longmapsto (q_1, \cdots, q_n)$

$\forall \varphi_x \in (T_x N)^* \longmapsto (q_1, \cdots, q_n, p_1, \cdots, p_n)$

if $\varphi_x = p_1 \, dq_1|_x + \ldots + p_n \, dq_n|_x$.

Take smallest topology on T^*N s.t. $\forall$ chart $\begin{array}{ccc} U & \to & \Omega \\ \cap & & \cap \\ N & & \mathbb{R}^n \end{array}$

$T^*N|_U$ is open

and the adapted coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$ are continuous.

atlas: $\left\{ (q_1, \cdots, q_n, p_1, \cdots, p_n) : T^*N|_U \longrightarrow \Omega \times \mathbb{R}^n \right\}$

Tautological 1-form on T^*N: $\qquad \alpha = \sum_{j=1}^n p_j \, dq_j$

Canonical sympl. form on T^*N: $\qquad \omega = -d\alpha = \sum_j dq_j \wedge dp_j$

in adapted coordinates.

independence of choice of coordinates:

$$\pi : \quad T^*N \longrightarrow N$$
$$\varphi_x \longmapsto x$$

$$\pi_* : \quad \zeta \longmapsto \pi_* \zeta$$
$$\zeta \in T_{\varphi_x}(T^*N), \qquad \pi_*\zeta \in T_x N$$

$$\alpha_{\varphi_x}(\zeta) := \varphi_x(\pi_* \zeta)$$

Natural.

$$\begin{array}{ccc} T^*N_1 & \xrightarrow[\text{its cotangent lift}]{\tilde{\varphi}} & T^*N_2 \\ \downarrow & & \downarrow \\ N_1 & \xrightarrow[\text{diffeo}]{\varphi} & N_2 \end{array}$$

Then: $\quad \tilde{\varphi}^* \alpha_2 = \alpha_1 \quad , \qquad \tilde{\varphi}^* \omega_2 = \omega_1$

tautological 1-forms on T^*N_1, T^*N_2.

Big open question (Eliashberg?): if T^*N_1 & T^*N_2 are symplectomorphic, are N_1 & N_2 are diffeomorphic?

$$G \circlearrowright T^*N \qquad \text{its cotangent lift}$$
$$\downarrow \qquad\qquad\qquad \Uparrow$$
$$G \circlearrowright N \qquad \text{Lie group action}$$

is Hamiltonian!

momentum map: $\mu : T^*N \longrightarrow \mathfrak{g}^*$

$\forall\, \xi \in \mathfrak{g}$ $\qquad \mu^\xi : T^*N \longrightarrow \mathbb{R}$

$\xi_N \in \mathfrak{X}(N)$, $\xi_{T^*N} \in \mathfrak{X}(T^*N)$ vector fields.

$$\boxed{\mu^\xi = -\alpha(\xi_{T^*N})}$$

tautological 1-form

Check Hamilton's equation:

$$d\mu^\xi = -d\,\alpha(\xi_{T^*N}) = -d\,\iota_{\xi_{T^*N}}\alpha \overset{\text{Cartan}}{=} -\underbrace{\boxed{\mathcal{L}_{\xi_{T^*N}}\alpha}}_{=0} + \iota_{\xi_{T^*N}} \overbrace{\boxed{d\alpha}}^{-\omega}$$

$=0$ because the lifted flow preserves α

So:
$$\boxed{d\mu^\xi = -\iota_{\xi_{T^*N}}\,\omega}$$

Hamilton's equation

More generally.

Take (M, ω) s.t. $\omega = -d\alpha$

ω: 2-form, α: 1-form.

$G \circlearrowright M$ that preserves α.

Then the action is Hamiltonian w/ momentum map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

$\forall \xi \in \mathfrak{g}$ $\qquad \mu^\xi : M \longrightarrow \mathbb{R}$

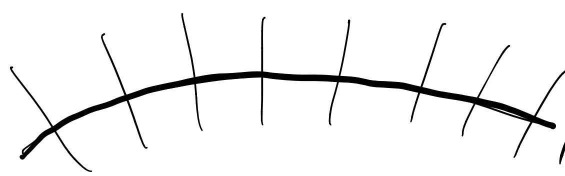
$$\mu^\xi = -\alpha(\xi_M) .$$

If ω is exact and G is compact and preserves ω

Then we can choose α to be G-invariant & do the same for this α.

Cotangent bundle.

dim $= 2n$



$N \approx$ the 0-section

dim $= n$

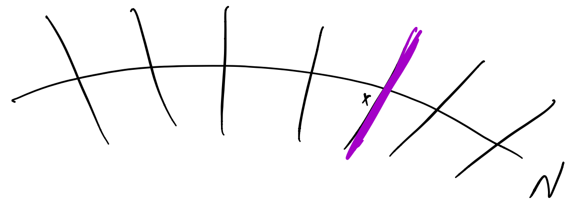
T^*N

In adapted coordinates: $p_1 = \dots = p_n = 0$.

$$\omega = \sum dq_j \wedge dp_j$$

$$i_0 : N \xrightarrow[\text{0-section}]{} T^*N$$

$$i_0^* \omega = 0 .$$



$i_x : \quad T_x^*N \hookrightarrow T^*N$

in adapted coordinates: $\quad q_1, \dots, q_n$ fixed

$p_1, \dots, p_n$ vary

$$i_x^* \omega = 0$$

a Lagragian submfld of a $2n$ dim'l sympl mfld (M, ω) is a submfld $N \xhookrightarrow{i} M$ s.t. $i^*\omega = 0$ and $\dim N = n$.

Special case of Alan Weinstein's local normal form: Lagr. tubular nbhd thm:

$\forall \quad N \xhookrightarrow{i} (M, \omega)$

embedded Lagr. submfld $\qquad$ sympl. mfld

$\exists \quad \begin{pmatrix} \text{nbhd} \\ \text{of } N \text{ in } M \end{pmatrix} \xrightarrow{\text{symplecto}} \begin{pmatrix} \text{nbhd} \\ \text{of 0-section} \\ \text{in } T^*N \end{pmatrix}$

s.t. $\quad x \longmapsto 0_x$

Recall ordinary tubular nbhd thm:

$$N \xhookrightarrow[\text{submfld}]{} M$$

Then

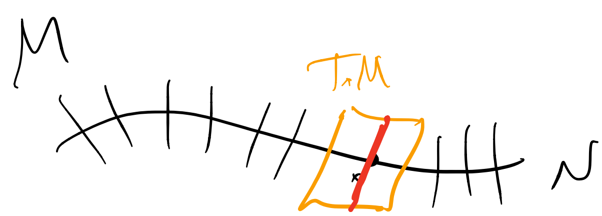
$$\begin{pmatrix} \text{nbhd of } N \\ \text{in } M \end{pmatrix} \xrightarrow{\exists \text{ diffeo}} \begin{pmatrix} \text{nbhd} \\ \text{of } 0\text{-section} \\ \text{in } \nu_M N \end{pmatrix}$$

$$x \longmapsto 0_x$$

The normal bundle of N in M:

$$\begin{array}{c} \nu_M N \\ \downarrow \\ N \end{array} = TM|_N \,/\, TN$$

$$\forall x \in N: \quad \left(\nu_M N\right)_x = T_x M / T_x N$$



Think of $\nu_M N$ as complementary to TN in $TM|_N$.

Sanity check for Weinstein's Lagr. tubular nd thm:

$$N \xhookrightarrow[\text{Lagr submd}]{} (M, \omega)$$

$$\nu_M N \overset{?}{\cong} T^*N \checkmark$$

$$\begin{tikzcd} TM|_N \arrow[r, "\omega^\#"', "\simeq"] \arrow[d] \arrow[rr, bend right] & (T^*M)|_N \arrow[r, "\text{restrict to } TN"] & T^*N \\ \nu_M N = (TM)|_N / TN \arrow[urr, "\cong"'] & & \end{tikzcd}$$

$v \mapsto \omega(v, \bullet)$

More about T^*N.

$\alpha :=$ tautological 1-form on T^*N

β any 1-form on N.

$\Longleftrightarrow$ section of $T^*N \longrightarrow N$

write:
$$\begin{tikzcd} T^*N \arrow[d] \\ N \arrow[u, bend left, "\beta"] \end{tikzcd}$$

Exercise $\beta^* \alpha = \beta$

tautological 1-form on T^*N

as a map from N to T^*N

as a 1-form on N

β 1-form on N

$\rightsquigarrow$ section $\beta : N \longrightarrow T^*N$

graph$(\beta) \subset T^*N$
submfld.

β is a closed 1-form iff graph$(\beta) \subset T^*N$ is a Lagr submfld.

proof: $d\beta = d\,\beta^*\alpha = \beta^* d\alpha = -\beta^*\omega$

as a 2-form on N

as a map $N \to T^*N$ which is a parametrization of graph(β).

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office hour. $G \circlearrowright M$

$x \in M$, $\operatorname{stab}(x) = H$.

linear isotropy $H \circlearrowright T_x M$.

claim. it preserves $T_x(G\cdot x)$.

In general. $\psi : M \to M$ diffeo

$N \subset M$ submfld, $\psi(N) = N$.

ψ restricts to $\psi|_N : N \to N$.

$x \in N$ s.t. $\psi(x) = x$.

$$\begin{array}{cccc} d\psi|_x : & T_x M & \longrightarrow & T_x M \\ & \cup & & \cup \\ & T_x N & \longrightarrow & T_x N \end{array}$$

Now take ψ = action by an element of H

$H \subset G \circlearrowright M$, x fixed by H.

$N = G\cdot x$. Then N is H-invariant.

so each element of H acts on $T_x M$ & preserves $T_x N$.