3/3/2021

we'll prove:

<u>Weinstein's tubular nbhd thm</u>

Let M be a mfld, $N \subset M$ a submfld,
ω_0, ω_1 closed 2-forms on nbhds of N in M
that agree and are nondegenerate along N.
i.e. as (fibrewise) sympl. structures on $TM|_N$.

Then $\exists$ open nbhds U_0 & U_1 of N in M
and $\exists$ diffeo $\psi : U_0 \longrightarrow U_1$ s.t.

$$\psi|_N = Id_N \quad \text{and} \quad d\psi|_N = Id_{TM|_N}$$

$$\text{and} \quad \psi^* \omega_1 = \omega_0 .$$

<u>This will imply</u>:

Weinstein's Lagrangian $\left\{ \begin{array}{l} \text{tubular nbhd thm:} \\ \text{normal form thm.} \end{array} \right.$ same thm, different names.

Let $N \hookrightarrow (M, \omega)$ be a Lagrangian submfld of a sympl. mfd. Then $\exists$ nbhd U of N in M and a nbhd V of the 0-section in T^*N and a symplectomorphism

$$(U, \omega) \longrightarrow (V, \omega_{can})$$

canonical s. form on T^*N

s.t. $\forall x \in N \qquad x \longmapsto O_x$

the origin in $T^*_x N$.

<u>We will need</u>:

Lemma: Let $N \hookrightarrow (M,\omega)$ be a Lagr. submfd.
Then $\exists$ Lagragian splitting $TM|_N = TN \oplus E$

i.e. $\exists$ Lagr subbundle E of $TM|_N$ that is complementary to TN.

More generally. a sympl. v. bundle over N is a v. bundle $W \to N$ equipped w/ a smooth fibrewise sympl. structure.

(thm: we can then choose the local trivializations of W to be (fibrewise) symplectic)

a Lagr subbundle is a subbundle
$$\begin{tikzcd} E' \arrow[r, hook] \arrow[dr] & W \arrow[d] \\ & N \end{tikzcd}$$
st. $\forall x \in N$ $E'_x \subset W_x$ is a Lagrangian subspace.

Lemma: Given such $E' \subset W$
$\exists$ another Lagr subbundle $E'' \subset W$ st $W = E' \oplus E''$.

Ordinary tubular nbhd thm.

Given a subnd $N \hookrightarrow M$ and a splitting $TM|_N = TN \oplus E$

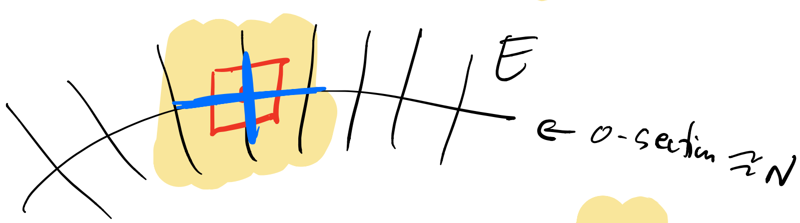
There exists a diffeo

$$\begin{pmatrix}\text{nbhd of } N \\ \text{in } M\end{pmatrix} \to \begin{pmatrix}\text{nd of} \\ \text{0-section} \\ \text{in } E\end{pmatrix}$$

s.t. $\quad x \longmapsto 0_x$

whose differential along N induces the identity map on $TN \oplus E$.

$$\begin{tikzcd} TM|_N \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & TE|_{\text{0-section}} \arrow[d, phantom, "\cong \text{ natural}"] \\ TN \oplus E & TN \oplus E \end{tikzcd}$$



using $T_{0_x} E_x \cong E_x \quad \forall x \in N$

(where 0_x = the origin in the v.space E_x)

and using $T_{0_x}(\text{0-section}) \cong T_x N$

and using $T_{0_x} E = T_{0_x}(\text{0-section}) \oplus T_{0_x}(E_x)$

Proof of Weinstein's Lagrangian normal form thm.

$$N \xhookrightarrow[\text{Lagr}]{} (M, \omega)$$

By the lemma $\exists$ Lagr splitting $TM|_N = TN \oplus E$.

$$\omega^\# : E \xrightarrow{\simeq} T^*N$$

Get $TM|_N \cong TN \oplus T^*N$ s.t.

$\forall x \in N$ is the standard sympl. str on $T_xN \oplus (T_xN)^*$:

$$\big((v_1, \varphi_1), (v_2, \varphi_2)\big) \longmapsto \varphi_2(v_1) - \varphi_1(v_2).$$

Note. $\forall$ v. bundle $E \to N$, $TE|_{0\text{-section}} \overset{\text{natural}}{\cong} TN \oplus E$

For $E = T^*N$: This isomorphism Takes $\omega_{can}|_{0\text{-section}}$ to the standard fibrewise s. str on $TN \oplus T^*N$.

Ordinary tubular nbhd thm gives a diffeo

$$\psi : \begin{pmatrix} \text{nbhd of } N \\ \text{in } M \end{pmatrix} \xrightarrow{\simeq} \begin{pmatrix} \text{nbhd of 0-section} \\ \text{in } T^*N \end{pmatrix}$$

$$x \longmapsto 0_x$$

whose differential along N induces the identity map on $TN \oplus T^*N$

so this diffeo is sympl. along the 0-section.

i.e. $(\psi^* \omega_{can})|_N = \omega|_N$ (on $TM|_N$).

Weinstein's tubular nbhd thm gives a symplecto

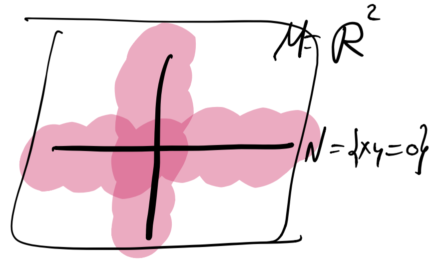
$$\left(\text{nbhd of } N,\ \psi^*\omega_{can}\right) \leftarrow \left(\begin{matrix}\text{nbhd}\\ \text{of } N\end{matrix},\ \omega\right)$$

Composing with ψ, we obtain a symplecto

$$\left(\text{nbhd of } N,\ \omega\right) \longrightarrow \left(\begin{matrix}\text{nbhd of}\\ \text{0-section in } T^*N\end{matrix},\ \omega_{can}\right)$$

Definition. $\underset{\text{set}}{N} \overset{i}{\hookrightarrow} V \underset{\text{open}}{\subset} \underset{\text{mfld}}{M}$

a smooth retraction of V to N is a smooth map $\pi : V \to N$ s.t. $\pi \circ i : N \to N$ is Id_N.



Example. $\nexists$ smooth retraction from a nbhd V of $\{xy=0\} =: N$ to $\{xy=0\}$

Proof. suppose $\pi : V \to V$ is smooth & satisfies $\pi \circ i = Id_N$.

Then $d\pi|_0 : R^2 \longrightarrow R^2$ is the identity map

so π is a local diffeo near 0

so $\pi(V)$ cannot be contained in N.

a smooth weak deformation retraction of V to N is a smooth map $R : [0,1] \times V \longrightarrow V$ s.t. the following hold. Let $r_t := R(t, \cdot) : V \longrightarrow V$.

- $r_1 = Id$
- $r_0(V) \subset N$
- $r_t(N) \subset N$ $\quad \forall t \in [0,1]$

"weak": $R_t(N)$ need not fix N.

("deformation retraction": would require $R_0|_N = Id_N$
"strong deformation retraction": would require $R_t|_N = Id_N$ $\forall t$).

Note: $R_0 : V \longrightarrow N$ is a smooth homotopy inverse to $i : N \hookrightarrow V$. Indeed,

$R_0 \circ i \simeq Id_N$ via $R_t \circ i : N \to N$ for $0 \le t \le 1$

$i \circ R_0 \simeq Id_V$ via $i \circ R_t : V \to V$ for $0 \le t \le 1$.

If M is a mfld and $N \subset M$ is an embedded submfd
Then $\forall$ nbhd of N in M contains an open nbhd V of N in M s.t. $\exists$ smooth strong deformation retraction of V to N. Indeed, the ordinary tubular nbhd thm identifies a nbhd of N in M w/ a starshaped nbd V of the 0-section of a v.bundle $\pi : E \to N$. Take $R_t : V \to V$ to be mult. by t.

Relative Poincaré lemma

Let $\underset{\text{set}}{N} \xhookrightarrow{i} V \underset{\text{open}}{\subset} M$.

Assume $\exists$ smooth weak deformation retraction of V to N

$$R : [0,1] \times V \longrightarrow V.$$

Fix such an R.
Let α be a closed k-form on V whose pullback to N vanishes: $i_N^* \alpha = 0$ explained below

Then $\exists$ $(k-1)$-form β on V whose pullback to N vanishes and s.t. $d\beta = \alpha$.

Moreover, we obtain a construction of such a β

s.t. $\alpha \longmapsto \beta$ satisfies

- If α vanishes along N, so does β.

  i.e. at the points of N

  i.e. on $TM|_N$.

- If $G \circlearrowright M$ preserves N & V
  and $R: [0,1] \times V \longrightarrow V$ is G-equivariant
  and α is G-invariant
  then β —— " ——.

- $\forall$ smooth family $\{\alpha_t\}_{t \in D}$ as above
  we obtain a smooth family $\{\beta_t\}_{t \in D}$ as above.

<u>Note</u>. If N is not a mfld

$i_N^* \alpha = 0$ means: $\forall p: U \longrightarrow V$

$U \subset \mathbb{R}^n$ open, $n = 1, 2, 3, \ldots$

if image $p \subset N$

Then $p^* \alpha = 0$.

Fibre integration (John Lee, ch. 17)

V mfld.

$\pi: [0,1] \times V \longrightarrow V$

$(t, x) \longmapsto x$

$\gamma \in \Omega^k([0,1] \times V)$: write $\gamma = \gamma_t' + \gamma_t'' \wedge dt$

where γ'_t, $0 \le t \le 1$, is a family of k-forms on V

& γ''_t, $0 \le t \le 1$ is a family of $(k-1)$-forms on V

push-forward $\equiv$ fibre integration

$$\pi_* : \Omega^k([0,1] \times V) \longrightarrow \Omega^{k-1}(V)$$

$$\gamma \longmapsto \int_0^1 \gamma''_t \, dt$$

Homotopy property: (proof: see John Lee Ch. 17)

Let $i_t : V \longrightarrow [0,1] \times V$ be

$x \longmapsto (t, x)$.

$$i_1^* \gamma - i_0^* \gamma = \pi_* d\gamma + d\pi_* \gamma$$

That is, π_* is a homotopy operator between i_0^* & i_1^* as morphisms of differential complexes $\Omega^*([0,1] \times V) \to \Omega^*(V)$

proof of the relative Poincaré Lemma

$\underset{\text{set}}{N} \overset{i}{\hookrightarrow} V \underset{\text{open}}{\subset} M$. $\quad R : [0,1] \times V \longrightarrow V$.

$R \circ i_1 = Id_V$, $\quad r : V \longrightarrow N$ is s.t. $i \circ r = \underbrace{R \circ i_0}_{\substack{V \to V \\ \text{image} \subset N}}$

$R_N := R|_{[0,1] \times N} : [0,1] \times N \longrightarrow N$.

$$\alpha = \alpha - \underbrace{r^* \alpha}_{\substack{=0 \\ \text{because } i_N^* \alpha = 0}} = i_1^* \underbrace{R^* \alpha}_{\gamma} - i_0^* \underbrace{R^* \alpha}_{\gamma} = \pi_* \underbrace{d(R^* \alpha)}_{= R^* d\alpha = 0} + d\pi_*(R^* \alpha)$$

$$\therefore \quad \alpha = d \underbrace{\pi_*(R^*\alpha)}_{=:\beta}$$

$$[0,1]\times V \xrightarrow{R} V$$
$$[0,1]\times V \xrightarrow{\pi} V$$

Variant on Weinstein's tubular nd thm.

$\underset{\text{set}}{N} \subset \underset{\text{mfld.}}{M}$

Assume each nbhd of N in M contains an open nd V of N in M s.t. $\exists$ smooth weak deformation retraction of V to N.

i.e. "N is a smooth weak nbhd deformation retract".

Let ω_0 & ω_1 be closed 2-forms on nbhds of N in M that agree & are nondeg along N. (i.e. on $TM|_N$).

Then $\exists$ open nbhds U_0 & U_1 of N in M and $\exists$ diffeo $\psi : U_0 \longrightarrow U_1$ s.t. $\forall x \in N$ $\psi(x) = x$ and $d\psi = Id$ on $TM|_N$

and $\psi^* \omega_1 = \omega_0$.