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submanifolds of symplectic mflds

Def

An immersion $f : N \hookrightarrow (M, \omega)$ is symplectic / isotropic / coisotropic / Lagrangian if $\forall x \in N$ $f_*(T_x N) \subseteq \left(T_{f(x)} M, \omega_{f(x)}\right)$ is a symplectic / isotropic / coisotropic / Lagrangian subspace.

a submfld $N \subseteq (M, \omega)$, or a weakly embedded submfld, is symplectic/isotropic/coisotropic/Lagrangian if its inclusion map $i : N \hookrightarrow M$ has this property.

N is a constant rank submfld if

$\omega_N := i^* \omega$,

which is a closed 2-form on N,

has constant rank:

i.e. $\omega_N^\# : TN \longrightarrow T^*N$ has the same rank at all pts of N.

Equivalently. $\mathrm{null}(\omega_N) := \ker \omega_N^\#$ has the same dimension at all pts of N.

null(ω_N) is the null distribution of (N, ω_N)

subbundle of TN.

Because ω_N is closed, null(ω_N) is involutive

i.e. $\forall$ two v.fields X, Y on N,

if X & Y are sections of null(ω_N),

so is $[X, Y]$.

(Recall. $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$

$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$

Lie derivative ($\mathcal{L}_X$)   Contraction (ι_Y))

proof that null(ω_N) is involutive:

Let X, Y be v.fields on N.

Assume $\iota_X \omega_N = 0$ and $\iota_Y \omega_N = 0$.

$$\iota_{[X,Y]} \omega_N = \mathcal{L}_X \iota_Y \omega_N - \iota_Y \underbrace{\mathcal{L}_X \omega_N}_{d\iota_X \omega_N + \iota_X d\omega_N} \qquad d\omega_N = 0$$

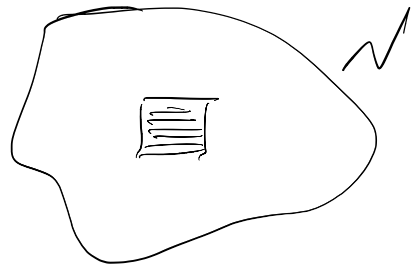
$$= \mathcal{L}_X \underbrace{\iota_Y \omega_N}_{=0} - \iota_Y d \underbrace{\iota_X \omega_N}_{=0}$$

$$= 0.$$

✓

By Frobenius's thm, null ω_N defines a foliation on N.

- $\exists$ Local foliation charts:



- N decompose into a disjoint union of connected weakly embedded submds ("leaves") s.t. at each pt x The tangent to the ~~leaf~~ leaf through x is $\text{null}(\omega_N|_x)$.

$\star$ See John Lee "Intro to smooth mflds".

This is the null foliation on N.

Define an equivalence relation on N: $x \sim y$ iff they're in the same leaf.

$\boxed{\text{Reduction}}$: $N_{red} := N/\sim$.

The foliation is fibrating if there exists a (necessarily unique) mfld structure on N_{red} s.t. The quotient map $\pi : N \twoheadrightarrow N_{red}$ is a submersion.

If so, there exists a unique 2-form ω_{red} on N_{red} s.t.

$$\pi^* \omega_{red} = \omega_N .$$

and ω_{red} is a nondegenerate and closed 2-form.

i.e. symplectic.

$\boxed{\text{Important special case}}$

$$G \circlearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}^*$$

connected compact Lie

sympl.

m. map

$N := \mu^{-1}(\{0\})$ is G-invariant.

Assume it's a regular level set of μ.

hence a mfld.

Lemma. (a) $G \circlearrowright N$ is locally free.

i.e. its stabilizers are discrete

(b) The G-orbits in N coincide w/ the leaves of the null foliation on N.

(Hence, N is a const-rank submfld.
In fact, N is a coisotropic submfld.)

If $G \circlearrowright N$ is free, then

$N_{red} := N/G$ is a mfld

(N_{red}, ω_{red}) the sympl. reduction of $G \circlearrowright (M,\omega) \to \mathfrak{g}^*$ at 0.

Example. $S^1 \circlearrowright \mathbb{C}^n = \mathbb{R}^{2n}$ $\quad z_j = x_j + i y_j = r_j e^{i\theta_j}$

$$\omega = \sum_j dx_j \wedge dy_j$$

$$= \sum_j \underbrace{\boxed{r_j\, dr_j}}_{ds_j} \wedge d\theta_j .$$

when all $z_j \neq 0$ $\quad$ where $s_j = \frac{r_j^2}{2}$

This action is generated by $\frac{\partial}{\partial\theta_1} + \dots + \frac{\partial}{\partial\theta_n}$.

Momentum map: $\mu(z_1, \dots, z_n) = \sum_{j=1}^{n} \frac{r_j^2}{2} - \frac{1}{2}$

$N := \mu^{-1}(\{0\}) = S^{2n-1}$ the unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$.

$N_{red} = S^{2n-1}/S^1 \underset{\text{diffeo}}{\cong} \frac{\mathbb{C}^n \setminus \{0\}}{\mathbb{C}^\times} =: \mathbb{C}P^{n-1}$ complex projective space

ω_{red} is the Fubini Study s. form on $\mathbb{C}P^{n-1}$.

$B^{2n-2} \subset \mathbb{C}^{n-1} = \mathbb{R}^{2n-2}$

unit ball. $= \{(z_1, \dots, z_{n-1}) \mid |z_1|^2 + \dots + |z_{n-1}|^2 < 1\}$

$$i : B^{2n-2} \hookrightarrow S^{2n-1} \subset \mathbb{C}^n$$

$$(z_1, \dots, z_{n-1}) \longmapsto \Big(z_1, \dots, z_{n-1}, \underbrace{\sqrt{1 - |z_1|^2 - \dots - |z_{n-1}|^2}}_{\neq 0}\Big)$$

Note $\underbrace{i^* \omega_{\mathbb{C}^n}}_{\sum_{j=1}^{n} dx_j \wedge dy_j} = \omega_{\mathbb{C}^{n-1}}\Big|_B$

$$\begin{tikzcd}
\left(B^{2n-2}, \omega_{\mathbb{C}^{n-1}}\right) \arrow[r, hook] \arrow[dr, "\substack{\text{sympl. embedding}\\ \text{w/ open dense image.}}"'] & \left(S^{2n-1}, (\omega_{\mathbb{C}^n})_{S^{2n-1}}\right) \arrow[d] \\
& \left(\mathbb{C}P^{n-1}, \underbrace{\omega_{red}}_{\text{Fubini Study}}\right)
\end{tikzcd}$$

In homog. complex coordinates, the image =

$$\{[z_1, \dots, z_n] \mid z_n \neq 0\}.$$

The earlier lemma follows from:

Duality properties of the momentum map

$$G \circlearrowright (M,\omega) \xrightarrow{\ \mu\ } \mathfrak{g}^* .$$

① $\quad \text{image } d\mu\big|_x = \mathfrak{h}^\circ \quad \text{in } \mathfrak{g}^*$

$\forall\, x \in M,\quad$ with $\mathfrak{h} := \mathrm{Lie}(\mathrm{Stab}(x))$
$\mathfrak{h}^\circ :=$ its annihilator in $\mathfrak{g}^*$

② $\quad \ker d\mu\big|_x = T_x(G\cdot x)^\omega \quad \text{in } T_xM .$

Equivalently: (1') $\quad \mathfrak{h} = \left(\text{image } d\mu\big|_x\right)^\circ \quad \text{in } \mathfrak{g}$

(2') $\quad T_x(G\cdot x) = \left(\ker d\mu\big|_x\right)^\omega \quad \text{in } T_xM .$

Corollary of (1'): If $d\mu\big|_x$ is onto, then $\mathfrak{h} = \{0\}$, so $H = \mathrm{Stab}(x)$ is discrete.

Corollary of (2'): Suppose $N := \mu^{-1}(\{0\})$ is a regular level set. Then $\forall x \in N$ $\quad T_xN = \ker d\mu\big|_x$.

By 2', $\quad (T_xN)^\omega = T_x(G\cdot x) \subset T_xN$.

So N is coisotropic, and its null distribution $\equiv$ the tangents to the G orbits.

↳ this proves the earlier lemma.

$\boxed{\text{proof of } 1'}$ Let $\xi \in \mathfrak{g}$.

$\xi \in (\text{image } d\mu|_x)^0$

$\Leftrightarrow \langle \underbrace{d\mu|_x(v)}_{\text{in } \mathfrak{g}^*}, \underbrace{\xi}_{\text{in } \mathfrak{g}} \rangle = 0 \quad \forall v \in T_x M$

$= d\mu^{\xi}|_x (v) = -\omega(\xi^\#, v)$

where $\xi^\#$ is the v.field corresponding to ξ

$\Leftrightarrow \omega(\xi^\#|_x, v) = 0 \quad \forall v$

$\Leftrightarrow \xi^\#|_x = 0$

$\Leftrightarrow \xi \in \text{Lie}(\text{stab}(x)) = \mathfrak{h}$.

Recall: For any action of a Lie group G on a mfd M: Let $\xi \in \mathfrak{g}$ & Let $\xi^\# \in \mathfrak{X}(M)$. Then $\forall x \in M$ $\text{Lie}(\text{stab}(x)) = \{\xi \mid \xi^\#|_x = 0\}$.

see e.g. John Lee intro to smooth mfds.

This uses: for $H \subset G$ $\mathfrak{h} = \{\xi \in \mathfrak{g} \text{ s.t. } \exp(t\xi) \in H \ \forall t \in \mathbb{R}\}$

$\boxed{\text{proof of } 2}$ Let $v \in T_x M$

$v \in \ker d\mu|_x$

$\Leftrightarrow \forall \xi \in \mathfrak{g} \quad d\mu^{\xi}|_x (v) = 0$

$\Leftrightarrow \forall \xi \in \mathfrak{g} \quad \omega_x(\underbrace{\xi^\#|_x}, v) = 0$

Recall $T_x(G \cdot x) = \{\xi^\#|_x \mid \xi \in \mathfrak{g}\}$.

$\Leftrightarrow v \in (T_x(G \cdot x))^{\omega}$

Reminder.

The standard Hermitian structure on $\mathbb{C}^n$ is

$$H(u,v) = \sum_{j=1}^{n} \overline{u_j}\, v_j$$

$$= \underbrace{g(u,v)}_{\text{the standard inner product on } \mathbb{R}^{2n}} + i\, \underbrace{\omega(u,v)}_{\text{The standard sympl. structure on } \mathbb{R}^{2n}}$$

On a real vector space V, a complex structure is an automorphism $J : V \to V$ s.t. $J^2 = -I$.

(it defines multiplication by $i = \sqrt{-1}$.)

On a real v. space V, consider

g an inner product

ω a symplectic structure

J a complex structure.

The compatibility conditions : $g(u,v) = \omega(u, Jv)$ $\forall u, v \in V$.

The standard g, ω, J on $\mathbb{C}^n$ are compatible.

Moreover, given any compatible g, ω, J on V, $\exists$ real linear isomorphism $V \cong \mathbb{C}^n = \mathbb{R}^{2n}$ that takes g, ω, J to the standard structures on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Sketch of Proof: Given compatible g, ω, J on V

J makes V into a complex v. space

Define $H(u,v) := g(u,v) + i\,\omega(u,v)$.

Then H is a Hermitian inner product on the cmplx space V.

an orthonormal basis. (exists by linear alg.)

identifies V with $\mathbb{C}^n$

and H w/ the std Herm. str on $\mathbb{C}^n$.

Note: $g(u,v) = \omega(u, Jv)$ implies that any two of g, J, ω determine the third.

Def. On a s.v. space (V, ω)

- a compatible inner product is an inner product g s.t. $\exists$ cplx str J s.t. $g(u,v) = \omega(u, Jv)$
- a compatible cplx str is a cplx str J s.t. $\exists$ inner product g s.t. $g(u,v) = \omega(u, Jv)$.

Nonlinear version

An almost complex structure on a mfld M is an automorphism $J : TM \to TM$ s.t. $J^2 = -I$

i.e. a cplx str J_x on each T_xM that varies smoothly in x.

If M is a complex mfld then $\forall x \in M$ T_xM is a cplx v. space so we get an almost cplx str

BUT most almost complex structures do not arise in this way.

For cplx mflds M, N $\quad f: M \to N$ is holomorphic

iff $\quad df|_x : T_x M \longrightarrow T_{f(x)} N$ is cplx linear $\forall x \in M$.

(By Cauchy-Riemann equation.)

For <u>almost</u> cplx mflds M, N

<u>define</u> $f: M \to N$ to be holomorphic

iff $\quad df|_x : T_x M \to T_{f(x)} N$ is cplx linear $\forall x \in M$.

"<u>Typically</u>" $\exists$ "many" holomorphic curves

$$f: \underbrace{\Sigma}_{\substack{\text{Riemann surface} \\ \text{e.g. open subset of } \mathbb{C}}} \longrightarrow M$$

"<u>Typically</u>" $\nexists$ holomorphic maps $f: M \to \mathbb{C}$.

Let (M, ω) be a sympl. mfld.

a <u>compatible</u> <u>almost</u> <u>cplx</u> <u>str</u> on M

is an almost cplx str $J: TM \to TM$, $J^2 = -I$

s.t. $\forall x \in M$ $\quad J_x$ is compatible w/ ω_x (on $T_x M$).

a Compatible <u>Riemannian metric</u> on M

is a R. metric g s.t. $\forall x \in M$ g_x is compatible w/ ω_x.

In either case we get a <u>compatible triple</u>

g, ω, J

$$g(u,v) = \omega(u, Jv) \qquad \begin{array}{l}\forall x \in M \\ \forall u, v \in T_x M.\end{array}$$

<u>Theorem</u>. Let (M,ω) be a sympl. mfld. Then it has a compatible almost complex str.

Moreover, Given $\underset{\text{compact Lie}}{G} \circlearrowright (M, \omega)$

$\exists$ G-invt compatible almost cplx str.

don't need $d\omega = 0$!

<u>Moreover</u>. Let $\begin{array}{c}(W,\omega) \\ \downarrow \\ N\end{array}$ be a symplectic vector bundle.

i.e. $\forall x \in N$ $\quad (W_x, \omega_x)$ is a sympl. v. space.

(Example $TM|_N$ if (M,ω) is sympl. and $N \subset M$ is a submfd)

Then $\exists$ fibrewise compatible complex structure $J : W \to W$, $J^2 = -I$.

Moreover, given $\begin{array}{ccc} G & \circlearrowright & (W,\omega) \\ \| & & \downarrow \\ \underset{\text{compact Lie}}{G} & \circlearrowright & N \end{array}$, $\exists$ such a J that is G-invariant.

proof: Next time.

Corollary. Let $\begin{matrix}(W,\omega)\\ \downarrow\\ N\end{matrix}$ be a sympl. vector bundle.

and let $L \subseteq W$ be a Lagrangian subbundle.
Then there exists a complementary Lagr. subbundle:

$$W = L \oplus E.$$

proof. Choose a fibrewise compatible cplx str J on W. Take $E = JL$.

Note an almost cplx str on a mfld M
$\Longleftrightarrow$ a fibrewise cplx str on the vector bundle $TM \to M$.

a compatible almost cplx str on a s. mfld (M,ω)
$\Longleftrightarrow$ a compatible fibrewise cplx str on the symplectic vector bundle (TM, ω).