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Compatible (almost) complex structure

Sympl. vector space (V, ω).

a compatible complex structure is

$$J : V \longrightarrow V \quad \text{s.t.} \quad J^2 = -I$$

& s.t. $g(u,v) := \omega(u, Jv)$

is an inner product

i.e. is symmetric and positive definite.

$\forall u \neq 0 \quad g(u,u) > 0$

Such a g is called a compatible inner product

Lemma If J is a compatible cplx str on (V, ω) and g is the corresponding inner product

Then

$$\omega(Ju, Jv) = \omega(u,v)$$

$$g(Ju, Jv) = g(u,v). \qquad \forall u, v$$

proof.

$$\omega(Ju, Jv) = g(Ju, v) = g(v, Ju)$$

$$= \omega(v, J^2 u) = \omega(v, -u) = \omega(u,v)$$

$g(Ju, Jv) = \ldots$ exercise.

Fix (V,ω).

$$\mathcal{J}(V,\omega) := \left\{ \begin{matrix}\text{compatible} \\ \text{complex structures}\end{matrix} \right\} \subset \{ L : V \to V \ \text{linear} \}$$

$$\mathcal{G}(V,\omega) := \left\{ \begin{matrix}\text{compatible} \\ \text{inner products}\end{matrix} \right\} \subset \{ B : V \times V \to \mathbb{R} \ \text{bilinear} \}$$

$$\begin{tikzcd} \mathcal{J}(V,\omega) \arrow[rr, leftrightarrow, "\text{bijection}"] & & \mathcal{G}(V,\omega) \\ J \arrow[u, phantom, sloped, "\in"] \arrow[rr, leftrightarrow] & & g \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

through $\boxed{g(u,v) = \omega(u, Jv)}$

$A \in Sp(V,\omega)$ acts on $\{L : V \to V\}$ & on $\{B : V \times V \to \mathbb{R}\}$

by

$$L \mapsto A_* L := A L A^{-1}$$

$$B \mapsto (A^{-1})^* B \ : \ (u,v) \mapsto B(A^{-1}u, A^{-1}v)$$

This action preserves $\mathcal{J}(V,\omega)$ & $\mathcal{G}(V,\omega)$.

and is transitive on each of them.

J, g, ω is a compatible triple iff $H(u,v) := g(u,v) + i\,\omega(u,v)$ is a Hermitian metric.

transitivity of the $Sp(V,\omega)$ action is a consequence of: $\forall$ Hermitian metric $\exists$ a basis in which it's standard.

$$B(u,v) = \omega(u, Lu)$$

defines an $Sp(V,\omega)$ – equivariant linear isomorphism, hence a ~~diffeomorphism~~

$$\left\{ \overset{\text{Linear}}{L : V \to V} \right\} \longleftrightarrow \left\{ \overset{\text{bilinear}}{B : V \times V \to \mathbb{R}} \right\}$$

That restricts to our bijection

$$\mathcal{J}(V,\omega) \longleftrightarrow \mathcal{G}(V,\omega)$$

hence, this bijection is an $Sp(V,\omega)$ – equivariant diffeo (of subsets of v. spaces).

not obvious

In fact, these sets are diffeomorphic to vector spaces. In particular, they are contractible submflds of the ambient v. spaces $\{L : V \to V\}$ & $\{B : V \times V \to \mathbb{R}\}$.

Conclusion. On any symplectic vector bundle

$$\begin{array}{c} W \\ \downarrow \\ N \end{array}$$

By applying the bijection $\mathcal{J}(V,\omega) \longleftrightarrow \mathcal{G}(V,\omega)$ for $V = W_x$ for each $x \in N$

We obtain a bijection

$(W,\omega) \downarrow N$

$$\left\{\begin{array}{c}\text{(smooth)} \\ \text{fibrewise} \\ \text{compatible} \\ \text{complex str's}\end{array}\right\} \overset{(*)}{\longleftrightarrow} \left\{\begin{array}{c}\text{(smooth)} \\ \text{fibrewise} \\ \text{compatible} \\ \text{inner products}\end{array}\right\}$$

$$J \longleftrightarrow g$$

$\forall x \in N$

$$\boxed{g(u,v) = \omega_x(u, J_x v)}$$

gives a well defined $J \Longleftrightarrow g$

Use local trivializations and smoothness of the bijection $\mathcal{J}(V,\omega) \longleftrightarrow \mathcal{G}(V,\omega)$ to show smoothness of $(*)$.

Recall For a subset X of $\mathbb{R}^N$

X is an k-dim'l embedded submfld iff

for each pt of X $\exists$ a diffeo

$$\underbrace{U}_{\substack{\text{a relative nbhd} \\ \text{of the point in } X \\ \text{wrt the subset topology}}} \longrightarrow \underbrace{\Omega}_{\substack{\text{an open subset} \\ \text{of } \mathbb{R}^k.}}$$

as subsets of $\mathbb{R}^N$ & $\mathbb{R}^k$.

a crucial lemma.

There exists a $Sp(V,\omega)$-equivariant smooth retraction

$$\underbrace{\left\{ \text{inner products on } V \right\}}_{\substack{\text{open subset of the v. space} \\ \text{of symmetric bilinear} \\ V\times V \to \mathbb{R}}} \longrightarrow \underbrace{\mathcal{J}(V,\omega)}_{\substack{\text{compatible} \\ \text{inner products} \\ \text{on } V}} \subset \{V\times V \to \mathbb{R}\}$$

Corollary. Given any sympl. v. bundle $\begin{matrix}(W,\omega) \\ \downarrow \\ N\end{matrix}$:

→ Choose any (smooth) fibrewise inner product

e.g. using local trivialization & patching w/ partition of unity.

→ apply the crucial lemma to obtain a (smooth) fibrewise compatible inner product

(In particular, every s. mfld has a compatible Riemannian metric.)

→ Take the corresponding compatible fibrewise cplx str.

(Thus, every s. mfld has a compatible almost cplx str).

Moreover, Given a G-equivariant sympl. vector bundle $G \circlearrowright \begin{matrix}(W,\omega) \\ \downarrow \\ N.\end{matrix}$

with G compact Lie:

→ choose any fibrewise inner product.

Average! To obtain a G-invariant fibrewise inner product.

Then the above procedure yields a <u>G-equivariant</u> compatible fibrewise cplx str on (W,ω)
$$\downarrow$$
$$N$$

<u>Thus.</u> Given $\underset{\text{Compact Lie}}{G} \circlearrowright \underset{\text{sympl. md}}{(M,\omega)}$

$\exists$ G-equivariant compatible almost cplx str $J : TM \longrightarrow TM$.

<u>Application</u> $G \circlearrowright (M,\omega)$ sympl. md

$N \subseteq M$ G-invt Lagr. submd

Then $\exists$ G-invt Lagrangian subbundle $E \subset \cancel{TN}\ TM|_N$

s.t. $TM|_N = TN \oplus E$.

(we used this in the proof of Weinstein's normal form for nbhds of Lagr. submflds)

<u>proof.</u> Let J be a G-invt compatible almost cplx str.

Define $E := JTN$. $\omega(Ju,Jv) = \omega(u,v) \Rightarrow E$ is Lagrangian

If $u \neq 0$, Ju is a friend of u:

$\omega(u, Ju) = g(u,u) > 0$

But TN is Lagrangian.

So $u \in TN$ has no friends in TN

So $TN \cap E = TN \cap J\,TN = \{0\}$.

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office hr. on $\mathbb{R}^{2n}$:

standard ω, J, g.

$$O(2n) = \{L : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid \forall u, v \in \mathbb{R}^{2n} \quad g(Lu, Lv) = g(u,v)\}$$

$$Sp(\mathbb{R}^{2n}) = \{L : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid \forall u, v \in \mathbb{R}^{2n} \quad \omega(Lu, Lv) = \omega(u,v)\}$$

$$GL_{\mathbb{C}}(n) = \{L : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \mid \forall u \quad L(Ju) = JL(u)\}$$

standard $H(u,v) = g(u,v) + i\,\omega(u,v)$

$$U(n) = \{L \in GL_{\mathbb{C}}(n) \mid \forall u, v \quad H(Lu, Lv) = H(u,v)\}$$

Facts

$$\begin{aligned} U(n) &= O(2n) \cap Sp(\mathbb{R}^{2n}) \cap GL_{\mathbb{C}}(n) \\ &= O(2n) \cap Sp(\mathbb{R}^{2n}) \\ &= O(2n) \cap GL_{\mathbb{C}}(n) \\ &= Sp(\mathbb{R}^{2n}) \cap GL_{\mathbb{C}}(n) \end{aligned}$$

This follows from $H(u,v) = g(u,v) + i\,\omega(u,v)$

and from $g(u,v) = \omega(u, Jv)$.

The std g on $\mathbb{R}^{2n}$ is repr. by the block matrix

$$\begin{array}{c} n \\ n \end{array} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

The std ω on $\mathbb{R}^{2n}$ is repr by

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(Check $\underbrace{(dx \wedge dy)}_{dx \otimes dy - dy \otimes dx}\big((a,b), (c,d)\big) = ad - bc$

$$= (a \ \ b) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$)

The std J on $\mathbb{R}^{2n}$ is repr by

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

(Check $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$

$J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$)