17/3/2021

welcome Back!

Assignment 3: 1-2 pages of math, in response to lectures.

Optional: 0-1 pages followup on earlier submission.

- ☆ Make it self contained
- ☆ Name & title e.g. \title{ } \section*{ }

12pt font.

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upcoming: proposed topics for presentation
aim for 2nd half of April.

+ written summaries

"Crucial lemma"

Fix a sympl. vector space (V,ω). Then $\exists$ a retraction

$$\begin{array}{ccc} \{\text{inner products on } V\} & \longrightarrow & \underbrace{\mathcal{J}(V,\omega)}_{\text{compatible inner products on } (V,\omega)} \\ \cap\! | & & \cap \\ \{V\times V \to \mathbb{R}\} & & \{V\times V \to \mathbb{R}\} \end{array}$$

that is smooth and $Sp(V,\omega)$-equivariant.

proof (by construction):

$$g \longmapsto \hat{g} : (u,v) \mapsto g\left(u, (-A^2)^{-\frac{1}{2}} v\right)$$

where $A := A_g : V \to V$ is s.t.

$$g(u,v) = \omega(u, Av)$$

some details. $A^T :=$ transpose of A wrt g.

Claim: $A^T = -A$

proof: $g(Au,v) = \omega(Au,Av) = \omega(-Av,Au) = g(-Av,u) = g(u,-Av)$

So $\underbrace{-A^2 = A^T A}$.

symmetric (wrt g): $(A^TA)^T = A^T(A^T)^T = A^TA$.

and positive definite (wrt g):

$$g(u, A^TAu) = g(Au, Au) > 0 \quad \forall u \neq 0$$

So we can take $(-A^2)^{-\frac{1}{2}}$

& it is symmetric & pos. def.

& commutes with A

i.e. $\hat{g}(u,v) = g(u, (-A^2)^{-\frac{1}{2}} v)$ is symmetric & pos. definite hence an inner product.

<u>Compatibility</u>: Let $J := A(-A^2)^{-\frac{1}{2}}$

$$\omega(u, Jv) = \omega(u, A(-A^2)^{-\frac{1}{2}} v) = g(u, (-A^2)^{-\frac{1}{2}} v) = \hat{g}(u,v)$$

$$J^2 = \left(A(-A^2)^{-\frac{1}{2}}\right)^2 = A^2\left((-A^2)^{-\frac{1}{2}}\right)^2 = A^2(-A^2)^{-1} = -I.$$

A commutes with $-A^2$, hence with $(-A^2)^{-\frac{1}{2}}$

$g \mapsto \hat{g}$ is a retraction: ~~If~~ g <u>is</u> compatible,

Then $A^2 = -I$ so $(-A^2)^{-\frac{1}{2}} = I$ & $\hat{g} = g$.

Prerequisites: bilinear forms & linear operators.

a bilinear form $B: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$
is represented by an $m \times m$ matrix M:

$$B(x, y) = x^T M y$$
$$= \langle x, My \rangle$$

- a bilinear form $B: V \times V \longrightarrow \mathbb{R}$
is symmetric if $\forall u, v \quad B(v,u) = B(u,v)$.
a symmetric bilinear form B is positive definite
if $\forall u \neq 0 \quad B(u,u) > 0$.

$$\{\text{inner products}\} := \left\{\begin{array}{c}\text{symmetric} \\ \text{pos. definite} \\ \text{bilinear forms}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{symmetric} \\ \text{bilinear} \\ V \times V \to \mathbb{R}\end{array}\right\}$$

open & convex.

- Given a fixed inner product $\langle , \rangle$ on V:
a linear operator $A: V \to V$ is symmetric
(wrt $\langle , \rangle$)

if the bilinear form $u, v \mapsto \langle u, Av \rangle$ is symmetric.
a symm. lin. operator A is positive definite
if this bilinear form is pos. definite.

A is <u>symmetric</u> iff it is diagonalizable and its eigenspaces are orthogonal wrt $\langle , \rangle$.

it is pos. definite if, additionally, the e.values are >0.

The <u>transpose</u> of A is A^T s.t. $\langle u, Av \rangle = \langle A^T u, v \rangle$ $\forall u, v$.

<u>More (expanded) prereqs</u>.

V real vector space.

$A : V \to V$ linear.

want to define: $\sqrt{A}$, $\exp A$, $\log A$, ...

$f(A)$ for f analytic.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function

$\Omega \subset \mathbb{C}$ Ω simply connected

$\overline{\Omega} = \Omega$

$f(\bar{z}) = \overline{f(z)} \quad \forall z \in \Omega$

eg. $f(z) = \exp(z)$ on $\mathbb{C}$

$f(z) = \log(z)$ on $\mathbb{C} \setminus (-\infty, 0]$ s.t. $\mathbb{R}_{>0} \to \mathbb{R}$

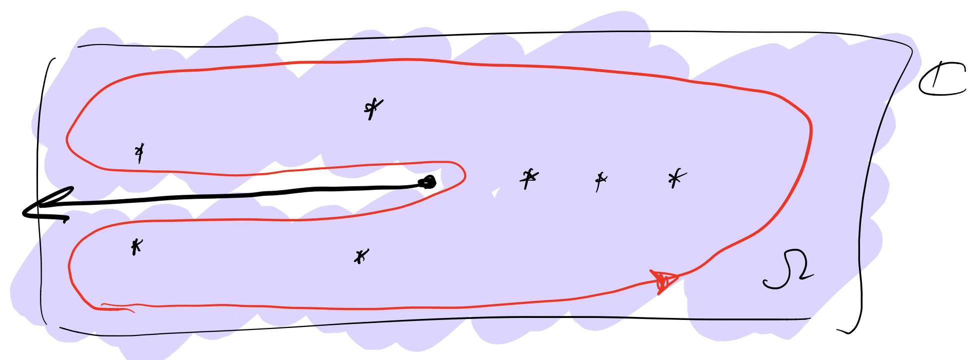
$f(z) = z^{1/2}$ on $\mathbb{C} \setminus (-\infty, 0]$ s.t. $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$

or z^{α} for any $\alpha \in \mathbb{R}$.

Assume $\underbrace{\text{Spec } A}$ $\subseteq \Omega$

$$:= \{ z \in \mathbb{C} \mid \nexists (zI - A)^{-1} \text{ on } V \otimes \mathbb{C} \}$$

example



Define $$f(A) := \frac{1}{2\pi i} \int_{z \in \Gamma} f(z) (zI - A)^{-1} \, dz$$

for Γ a counterclockwise closed path in Ω that surrounds $\text{spec}(A)$.

- It's independent of Γ
- $f(A)$ is an operator on V (not just on $V \otimes \mathbb{C}$)
- If A is diagonalizable with distinct e.values $\lambda_1, \cdots, \lambda_m$ (on V or on $V \otimes \mathbb{C}$)

Then $f(A)$ is diagonalizable w/ e.values $f(\lambda_1), \cdots, f(\lambda_m)$ on same e. spaces

Details. write $V = \bigoplus_{\lambda} V_{\lambda}$ s.t. $A\big|_{V_\lambda} = \lambda I_{V_\lambda}$.

Let $\boxed{v \in V_\lambda}$. Then $f(A)(v) = \frac{1}{2\pi i} \int_{z\in\Gamma} f(z) \big(\underset{\text{operator}}{(zI-A)^{-1}} \underset{\text{vector}}{(v)} \big) \, dz$

$$= \frac{1}{2\pi i} \int_{z\in\Gamma} f(z) \cdot \underset{\text{scalar}}{\frac{1}{z-\lambda}} \cdot v \; dz = \underbrace{\left(\frac{1}{2\pi i} \int_{z\in\Gamma} \frac{f(z)}{z-\lambda} \, dz \right)}_{\text{scalar}} v$$

$$= f(\lambda)$$

- $\forall$ operator C $\quad f(CAC^{-1}) = C f(A) C^{-1}$
- If C commutes with A
  Then C commutes with $f(A)$.
- If $f(z) = \sum a_n z^n$ on Ω
  Then $f(A) = \sum a_n A^n$

  eg. $\exp(A) = \sum A^n / n!$

  $\exp(-A) = \exp(A)^{-1}$

- $A \mapsto f(A)$ is smooth

$$\left\{ \begin{matrix} \text{operators } V \to V \\ \text{with spectrum} \subset \Omega \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{operators} \\ V \to V \end{matrix} \right\}$$

$$(A, \alpha) \longmapsto A^{\alpha} \quad \text{is smooth}$$

$$\left\{ \begin{array}{l} \text{operators } V \to V \\ \text{w/ spectrum} \subset \mathbb{C} \setminus (-\infty, 0] \end{array} \right\} \times \mathbb{R} \to \left\{ \begin{array}{c} \text{operators} \\ V \to V \end{array} \right\}.$$

- $f(A^T) = f(A)^T$
- If A is symmetric, so is $f(A)$
- If A is symmetric & positive definite and if $\forall x > 0$ $f(x) > 0$ Then $f(A)$ is also symm. & pos. def.

for any fixed inner product $\langle , \rangle$ on V.

Lemma. Fix $(\mathbb{R}^{2n}, \omega)$ ← standard s. str.

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \omega(u, v) = u^T \Omega v.$$

$$A \in Sp(\mathbb{R}^{2n}) \quad \text{iff} \quad A^T = \Omega A^{-1} \Omega^{-1}$$

$$\alpha \in sp(\mathbb{R}^{2n}) \quad \text{iff} \quad \alpha^T = \Omega(-\alpha)\Omega^{-1}$$

proof. $A \in Sp(\mathbb{R}^{2n})$ iff $\forall u, v$ $\underbrace{\omega(Au, Av) = \omega(u, v)}$.

This $\Longleftrightarrow$ $\underbrace{(Au)^T \Omega A v}_{u^T A^T \Omega A v} = u^T \Omega v$

So $A \in Sp(\mathbb{R}^{2n})$ iff $\underbrace{A^T \Omega A = \Omega}$

this $\Longleftrightarrow$ $A^T = \Omega A^{-1} \Omega^{-1}$.

The 2nd part is similar. It uses:
$\alpha \in sp(\mathbb{R}^{2n})$ iff $\omega(\alpha u, v) + \omega(u, \alpha v) = 0$ $\forall u, v$.

Polar decomposition of matrices.

Consider $\mathbb{R}^m$, standard $\langle \, , \rangle$.

$$\left\{\begin{matrix}\text{symmetric}\\ \text{pos. definite}\\ \text{matrices}\end{matrix}\right\} \times O(k) \longleftrightarrow GL(\mathbb{R}^k)$$

$$B, C \longmapsto A := BC$$

$$B := (A^T A)^{1/2} \longleftarrow A$$
$$C := B^{-1} A$$

$\boxed{\text{Assume } k = 2n}$ Claim: $A \in Sp(\mathbb{R}^{2n})$ iff $B, C \in Sp(\mathbb{R}^{2n})$

(This follows from the criteria (A))

Since $O(2n) \cap Sp(\mathbb{R}^{2n}) = U(n)$, the polar decomposition gives:

$$\left\{\begin{matrix}\text{symmetric}\\ \text{positive definite}\\ \text{matrices}\\ \text{in } Sp(\mathbb{R}^{2n})\end{matrix}\right\} \times U(n) \longleftrightarrow Sp(\mathbb{R}^{2n})$$

it's right-$U(n)$-equivariant.

It descends to a diffeomorphism:

$$\left\{\begin{matrix}\text{symmetric}\\ \text{pos. definite matrices}\\ \text{in } Sp(\mathbb{R}^{2n})\end{matrix}\right\} \longleftrightarrow Sp(\mathbb{R}^{2n}) / U(n)$$

RHS: quotient mfld structure

LHS: subset of $\mathbb{R}^{2n \times 2n}$

$\therefore$ the LHS <u>is</u> a submfld of $\mathbb{R}^{2n \times 2n}$.

The LHS:

$$\mathrm{Sp}(\mathbb{R}^{2n}) / U(n) \longleftrightarrow \{(V, \omega) \text{ compatible cplx strs.}\}$$

via $A \longmapsto A J_0 A^{-1}$

where J_0 is the std cplx str. represented by $-\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

& where $U(n)$ is wrt the std identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$

The RHS:

$$\left\{ \begin{array}{c} \text{symmetric positive definite matrices} \\ \text{in } \mathrm{Sp}(\mathbb{R}^{2n}) \end{array} \right\}$$

$$\cap$$

$$\left\{ \begin{array}{c} \text{symmetric} \\ \text{positive definite matrices} \end{array} \right\} \underset{\log}{\overset{\exp}{\rightleftarrows}} \left\{ \begin{array}{c} \text{symmetric} \\ \text{matrices} \end{array} \right\}$$

$$\overset{\Psi}{A} \longleftrightarrow \overset{\Psi}{\alpha}$$

Claim: A is in $Sp(\mathbb{R}^{2n})$
iff α is in $\mathfrak{sp}(\mathbb{R}^{2n})$

proof uses the criteria ☆

$\therefore$

$$\left\{ \begin{array}{c} \text{symm.} \\ \text{pos. def. matrices} \\ \text{in } Sp(\mathbb{R}^{2n}) \end{array} \right\} \underset{\exp}{\overset{\log}{\rightleftarrows}} \left\{ \begin{array}{c} \text{symm.} \\ \text{matrices} \\ \text{in} \\ \mathfrak{sp}(\mathbb{R}^{2n}) \end{array} \right\}$$

a vector subspace of $\{V \to V\}$

$\updownarrow$

$Sp(V,\omega)/U(n)$

$\updownarrow$

$\mathcal{J}(V,\omega)$

$\therefore$ $\mathcal{J}(V,\omega)$ is diffeomorphic to a vector space.

New topic ↓

Hamiltonian symplectomorphisms.

(M, ω) sympl. mfld.

a Hamiltonian isotopy from ψ_a to ψ_b

is a smooth family of diffeomorphisms

$$\psi_t : M \longrightarrow M \quad , \quad t \in [a, b]$$

s.t. $\exists$ time-dependent Hamiltonian $H_t : M \to \mathbb{R}$, $t \in [a,b]$

i.e. smooth family of functions

s.t. the time-dependent vector field X_t s.t. $\imath(X_t)\,\omega = dH_t$

generates the isotopy: $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

usually $a = 0$ and $\psi_0 = Id_M$.