19/3/2021.

Hamiltonian isotopies

Let (M,ω) be a symplectic mfld.

a smooth family of diffeomorphism $\psi_t : M \to M$, $t \in [a,b]$,

soon to be: symplectomorphisms, usually with $\psi_a = Id$ when $t=a$.

is a Hamiltonian isotopy if

$\exists$ smooth family of fctns

$H_t : M \to \mathbb{R}$, $t \in [a,b]$

s.t. the time-dependent v.field X_t

defined by $\iota_{X_t} \omega = -dH_t$ satisfies $\frac{d}{dt}\psi_t(x) = X_t \circ \psi_t$

$\forall t \in [a,b]$.

With this (nonstandard) definition,
For any diffeomorphism $f : M \to M$,
if $\{\psi_t\}$ is a Hamiltonian isotopy, so is $\{\psi_t \circ f\}$.

Lemma. For such $\{\psi_t\}$, if $\exists t_0$ s.t. ψ_{t_0} is a symplectomorphism
then $\forall t$ ψ_t is a symplectomorphism.

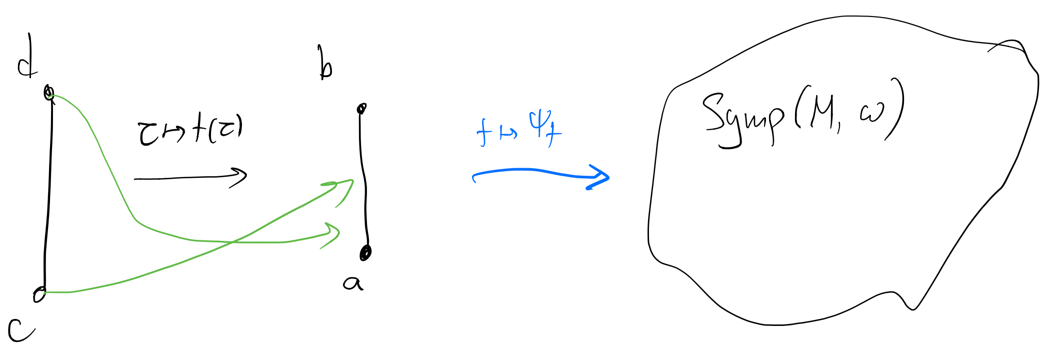
Proof. $\frac{d}{dt}\psi_t^* \omega = \psi_t^* \underbrace{\mathcal{L}_{X_t} \omega}_{\iota_{X_t} \underbrace{d\omega}_{0} + d \underbrace{\iota_{X_t} \omega}_{-dH_t} = 0+0} = 0$

so $\forall t$ $\psi_t^* \omega = \psi_{t_0}^* \omega = \omega$. ✓

Reparametrization

If $\{\psi_t\}$ is a Hamiltonian isotopy generated by $\{H_t\}$ and $t = t(\tau)$ is a smooth function

then $\hat{\psi}_\tau := \psi_{t(\tau)}$ is a Hamiltonian isotopy generated by $\hat{H}_\tau := H_{t(\tau)} \cdot \frac{dt}{d\tau}$



proof. $$d\hat{H}_\tau = \frac{dt}{d\tau} \cdot dH_{t(\tau)} = -\frac{dt}{d\tau} \cdot \iota_{X_{t(\tau)}} \omega$$

where $\frac{d}{dt}\psi_t = X_t \circ \psi_t$

$$= -\iota_{\hat{X}_\tau} \omega \qquad \text{where} \quad \hat{X}_\tau = \frac{dt}{d\tau} X_{t(\tau)}$$

and $$\frac{d}{d\tau}\hat{\psi}_\tau = \frac{d}{d\tau}\psi_{t(\tau)} = \frac{dt}{d\tau} \cdot \frac{d}{dt}\Big|_{t=t(\tau)} \psi_t$$

$$= \underbrace{\frac{dt}{d\tau} \cdot X_{t(\tau)}} \circ \psi_{t(\tau)} = \hat{X}_\tau \circ \hat{\psi}_\tau$$

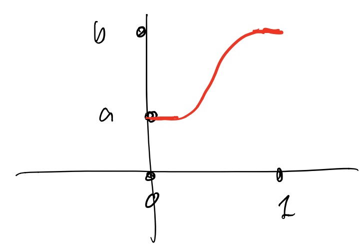
Corollary

If $\exists$ a Hamiltonian isotopy $\{\psi_t\}_{t \in [a,b]}$ from $\psi = \psi_a$ to $\psi' = \psi_b$

then $\exists$ Hamiltonian isotopy $\{\hat{\psi}_\tau\}_{\tau \in [0,1]}$ from ψ to ψ'

generated by $\{\hat{H}_\tau\}$ s.t. $\hat{H}_\tau \equiv 0$ for τ near 0 and for τ near 0.

pf. Reparametrize by



Concatenation

Let $\{\psi_t^1\}_{t \in [0,1]}$ be a Hamiltonian isotopy from ψ to ψ' generated by $\{H_t^1\}$ s.t. $H_t^1 = 0$ for t near 0 & t near 1

& Let $\{\psi_t^2\}_{t \in [0,1]}$ be a Hamiltonian isotopy from ψ' to ψ'' gen. by $\{H_t^2\}$ s.t. $H_t^2 = 0$ for t near 0 & t near 1.

Then $\psi_\tau := \begin{cases} \psi^1_\tau & \text{for } 0 \le \tau \le 1 \\ \psi^2_{\tau-1} & \text{for } 1 \le \tau \le 2 \end{cases}$

is a Hamiltonian isotopy from ψ to ψ''

gen. by $H_\tau := \begin{cases} H^1_\tau & \tau \in [0,1] \\ H^2_{\tau-1} & \tau \in [1,2] \end{cases}$.

The Hamiltonian group is

$$\mathrm{Ham}(M,\omega) := \left\{ \psi \;\middle|\; \begin{array}{l} \exists \text{ Hamiltonian isotopy} \\ \text{from Id to } \psi \end{array} \right\}.$$

$\rightarrow$ $\mathrm{Ham}(M,\omega) \subseteq \mathrm{Symp}(M,\omega)$.

$\longrightarrow$ $\mathrm{Ham}(M,\omega)$ is a group:

If $\{\varphi_t\}_{t\in[0,1]}$ is a Hamiltonian isotopy from $\varphi_0 = \mathrm{Id}$ to $\varphi_1 = \varphi$

Then $\varphi_{1-t} \circ \varphi^{-1}$ is a Hamiltonian isotopy from Id to φ^{-1}.

If also $\{\psi_t\}_{t\in[0,1]}$ is a Hamiltonian isotopy from $\psi_0 = \mathrm{Id}$ to $\psi_1 = \psi$

and $\{\varphi_t\}, \{\psi_t\}$ are generated by Hamiltonian functions that are $\equiv 0$ near $t=0, t=1$

then
$$\tilde{\psi}_\tau := \begin{cases} \psi_\tau & \tau \in [0,1] \\ \varphi_{\tau-1} \circ \psi_1 & \tau \in [1,2] \end{cases}$$

is a Hamiltonian isotopy from Id to $\varphi \circ \psi$.

Let $(M, \omega = d\alpha)$ be an exact symp. mfld.

Let $\psi_t : M \to M$ be s.t. $\psi_t^* \alpha = \cancel{0}\ \alpha$

(so that $\psi_t^* \omega = \omega$)

Let $H_t = + \iota_{X_t} \alpha$ where $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

$$dH_t = + d\iota_{X_t} \alpha = - \iota_{X_t} d\alpha = - \iota_{X_t} \omega$$

$$0 = \frac{d}{dt} \psi_t^* \alpha = \psi_t^* \mathcal{L}_{X_t} \alpha \quad \text{so} \quad \mathcal{L}_{X_t} \alpha = 0$$

So: ψ_t is a Hamiltonian isotopy.

Arnold's conjecture

Let (M, ω) be a compact symplectic mfld.

Let $\psi \in Ham(M, \omega)$. Then (conjecture:)

$$\# \{ x \mid \psi(x) = x \} \geq Crit\ M$$

where $Crit\ M := \min_{f \in C^\infty(M)} \# Crit\ f$

where $Crit\ f = \{ x \mid df|_x = 0 \}$

Note ~~If $\dim M \geq 2$~~ If M is not empty and not a singleton then $Crit\ M \geq 2$.

Nonexample. $M = \mathbb{R}^2 / \mathbb{Z}^2$ with $dx \wedge dy$

Then nontrivial translations are (non Hamiltonian) symplectomorphisms without fixed points.

office hr.

$$\iota_{X_t}\omega = -dH_t$$

$$v \longleftrightarrow \alpha$$

$$\alpha = \iota_v \omega$$

$$\begin{tikzcd} TM \arrow[r, bend left, "\omega^{\#}"] \arrow[d] & T^*M \arrow[l, bend left, "(\omega^{\#})^{-1}"] \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd}$$

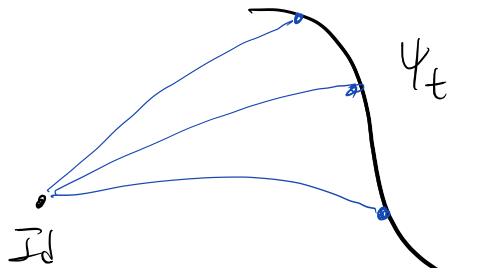
$\Rightarrow$

$$\begin{tikzcd} \mathfrak{X}(M) \arrow[r, bend left, "\omega^{\#}"] & \Omega^1(M) \arrow[l, bend left, "(\omega^{\#})^{-1}"] \end{tikzcd}$$

also: $\underset{\text{family of v.fields}}{\text{smooth}} \longleftrightarrow \underline{\text{smooth}}$ family of 1-forms.

<u>Note</u> Assume $\{\varphi_t\}$ is a Hamiltonian isotopy and $\exists t_0$ s.t. $\varphi_{t_0} \in Ham(M,\omega)$. Then $\forall t$ $\varphi_t \in Ham(M,\omega)$.

<u>Thm</u> (Banyaga): Let $\{\varphi_t\}$ be a smooth family of diffeo. assume $\forall t$ $\varphi_t \in Ham(M)$. Then $\{\varphi_t\}$ is a Hamiltian isotopy.



$\mathrm{Ham}(M,\omega)$ acts transitively on connected components:

$\mathbb{R}^{2n}$