24/3/2021

- HW #3: math.
- Topic for final projects
  - will post tentative list
  - talk to me!

"Baby Arnold Conjecture". — proved by Alan Weinstein.

Let (M,ω) be a compact symplectic mfld that is not a singleton, with $H^1(M)=0$.

Then $\exists$ a C^1 - neighborhood Ω of Id in $Symp(M,\omega)$ s.t.

I didn't show this part

$\boxed{1}$ $\Omega \subset Ham(M,\omega)$

in particular, $Symp(M,\omega)$ is locally connected

$\boxed{2}$ $\forall\ \psi : M \longrightarrow M$ in Ω

$$\#\{x \mid \psi(x) = x\} \geq \underbrace{Crit\ M}_{:= \min_{f\in C^\infty(M)} \underbrace{Crit\ f}_{\#\{x \mid df|_x = 0\}}}$$

The C^0 topology is the compact-open topology.

The C^1 topology . on $\{\text{smooth maps } M \to M\}$.

$$\left\{ \begin{matrix} \text{smooth} \\ \psi : M \to M \end{matrix} \right\} \xrightarrow{\qquad} \left\{ \begin{matrix} \text{smooth} \\ TM \to TM \end{matrix} \right\}$$

$$\psi \mapsto d\psi$$

& take the subspace topology induced from the compact-open topology on $\{TM \to TM\}$.

Note: the C^1 topology contains the C^0 topology.

Compact-open topology: "convergence" means pointwise convergence that is uniform on compact subsets.

Fix a chart on M: $\varphi = (x_1, \cdots, x_n) : U \longrightarrow \Omega$

$$\begin{array}{ccc} U & & \Omega \\ \cap & & \cap \\ M & & \mathbb{R}^k \end{array}$$

Fix $K \subset U$ compact

Fix $\varepsilon > 0$

$k = 2n$

$$\left\{ \psi : M \to M \;\middle|\; \begin{array}{l} \psi(K) \subset U \\ \psi = (\psi_1(\underline{x}), \cdots, \psi_k(\underline{x})) \text{ satisfies} \\ |\psi_j(\underline{x}) - x_j| < \varepsilon \quad \text{and} \quad \left| \dfrac{\partial \psi_j}{\partial x_j}(\underline{x}) - \delta_{ij} \right| < \varepsilon \\ \forall i, j \end{array} \right\}$$

is a C^1 nbhd of Id.

Finite intersections of such sets form a local base to the C^1 topology at Id.

A similar description holds for C^1 nbhds of maps $M \to M$ other than Id.

Crucial fact.

Assume M is compact.

$$\left\{ \begin{array}{c}\text{diffeomorphisms} \\ M \to M\end{array} \right\} \subset \left\{ \begin{array}{c} C^\infty \text{ maps} \\ M \to M \end{array} \right\}$$

is open in the C^1 topology.

Reference: book "Differential topology", Hirsch.

Important Lagrangian submds.

$\boxed{1}$ (M, ω) symplectic.

$\psi : M \to M$ diffeomorphism

$$\operatorname{graph} \psi = \{ (x, \psi(x)) \} \subset (\overline{M} \times M, -\omega \oplus \omega)$$

ψ is a symplectomorphism iff $\operatorname{graph} \psi \subset \overline{M} \times M$ is Lagrangian

Note graph ψ is parametrized by M via $x \mapsto (x, \psi(x))$

Proof. graph ψ is Lagr

iff $$\underbrace{\left(x \mapsto (x, \psi(x))\right)^* (-\omega \oplus \omega)}_{= \psi^* \omega - \omega} = 0$$

iff $\psi^* \omega = \omega$.

$\boxed{2}$ $\beta \in \Omega^1(M)$

$$\text{graph } \beta = \{ \beta_x \mid x \in M \} \subset (T^*M, \omega_{can})$$

$d\beta = 0$ iff graph $\beta \subset T^*M$ is Lagrangian.

proof. Recall $\omega_{can} = -d\alpha_{taut}$

graph β as a subset of T^*M

is parametrized by

$$\begin{tikzcd} \sigma_\beta : M \arrow[r] & T^*M \\ x \arrow[r, mapsto] & \beta_x \end{tikzcd}$$

so graph β is Lagr iff $\sigma_\beta^* \omega_{can} = 0$

But $\sigma_\beta^* \alpha_{taut} = \beta$

so $d\beta = d\sigma_\beta^* \alpha_{taut} = \sigma_\beta^* d\alpha_{taut} = -\sigma_\beta^* \omega_{can}$

so $d\beta = 0$ iff $\sigma_\beta^* \omega_{can} = 0$

Note. $\boxed{1'}$ $\psi : M \to M$ symplecto.

$$\{x \mid \psi(x) = x\} \xleftrightarrow{\text{bijection}} \Delta \cap \text{graph } \psi$$

$\Delta = \{(x,x)\}$

graph $\psi = \{(x, \psi(x))\}$

$\boxed{2'}$ $\beta \in \Omega^1(M)$ exact: $\beta = df$.

$$\mathrm{Crit}\, f \xleftrightarrow{\text{bijection}} 0_M \cap \operatorname{graph} \beta$$

$$0_M = \{0_x \in T^*_x M\} \subset T^*M$$

zero section

$$\operatorname{graph} \beta = \{\beta_x\}$$

Proof of "Baby Arnold" (M,ω) compact, $H^1(M) = 0$

$$M \xhookrightarrow[x \mapsto (x,x)]{} \tilde{M} \times M \qquad \text{image} =: \Delta \text{ the diagonal}$$

$$M \xhookrightarrow[x \mapsto 0_x]{} T^*M \qquad \text{image} =: 0_M \text{ the zero section}$$

} Lagr. embeddings.

$$\begin{tikzcd}
 & \Delta \arrow[r, phantom, "\subset"] & \mathcal{O}_\Delta \arrow[r, phantom, "\underset{\text{open}}{\subset}"] \arrow[dd, "\text{Weinstein local normal form for nbhds of Lagrangians}"] & \tilde{M} \times M \\
M \arrow[ur, "x \mapsto (x,x)"] \arrow[dr, hook, "x \mapsto 0_x"'] \arrow[r, phantom, "\text{diagram commutes}"] & & & \\
 & 0_M \arrow[r, phantom, "\subset"] & \mathcal{O}_0 \arrow[r, phantom, "\underset{\text{open}}{\subset}"] & T^*M
\end{tikzcd}$$

$\{\psi : M \xrightarrow{\text{smooth}} M \mid \psi \text{ is a diffeomorphism}\}$ is C^1 open in $\left\{ \overset{C^\infty \text{ maps}}{M \to M} \right\}$

$\{\psi : M \xrightarrow{\text{smooth}} M \mid \operatorname{graph} \psi \subset \mathcal{O}_\Delta\}$ is C^0 open, hence C^1 open in $\left\{ \overset{C^\infty \text{ maps}}{M \to M} \right\}$

For ψ in the second of these sets

$$\psi \longmapsto \psi^\# : M \longrightarrow \mathcal{O}_\odot$$

$$x \longmapsto (x, \psi(x))$$

$$\begin{tikzcd}
 & & T^*M \arrow[dr, "\pi"] & \\
M \arrow[r, "\text{graph}"'] \arrow[rr, bend right=40, "\psi^\#"'] & \mathcal{O}_\Delta \arrow[r, "\text{Weinstein}"'] & \mathcal{O}_\odot \arrow[r] \arrow[u, phantom, sloped, "\subset"] & M
\end{tikzcd}$$

$$\left\{ \psi \;\middle|\; \begin{array}{l} \text{graph } \psi \subset \mathcal{O}_\Delta \\ \text{and} \quad \pi \circ \psi^\# : M \to M \text{ is a diffeo} \end{array} \right\}$$

is a C^1 open nbhd of Id in $\{M \to M\}$.

For such ψ, ψ in this C^1 nbhd of Id

$$\beta_\psi := \psi^\# \circ (\pi \circ \psi^\#)^{-1} : M \longrightarrow T^*M$$

is a section

i.e. β_ψ is a 1-form.

ψ is a symplecto

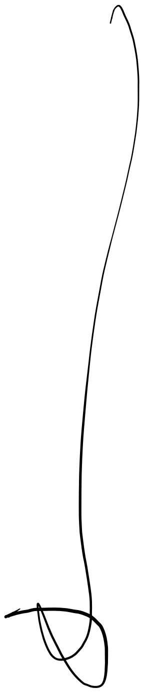
$\iff$ graph ψ is Lagr

$\iff$ β_ψ is closed

$\iff$ β_ψ is exact : write $\beta_\psi = df$

If so, $\{x \mid \psi(x) = x\} \xleftrightarrow{\text{bijection}} \text{Crit } f.$

So $\# \{x \mid \psi(x) = x\} = \text{Crit } f \geq \text{Crit } M.$



Poisson Bracket

(M,ω) sympl. mfld.

$$\begin{tikzcd}
TM \arrow[r, leftrightarrow] & T^*M \\
u \arrow[r, leftrightarrow] & \alpha = \omega(u,\bullet) \\
v \arrow[r, leftrightarrow] & \beta = \omega(v,\bullet)
\end{tikzcd}$$

$\omega : \underbrace{TM \times TM}_{\text{fibrewise product}} \longrightarrow \mathbb{R}$ 2-form i.e. section of $\Lambda^2 T^*M$

$\Longrightarrow \quad \pi : \underbrace{T^*M \times T^*M}_{\text{fibrewise product}} \longrightarrow \mathbb{R}$ bivector field i.e. section of $\Lambda^2 TM$

$$\pi(\alpha,\beta) := \omega(u,v) \qquad \forall\, \alpha,\beta \in T_x M$$

$$= \alpha(v) = -\beta(u) \qquad \forall\, x \in M$$

Today's sign convention for Hamilton's equation

$$df = \iota(\xi_f)\,\omega$$

so.

$$\xi_f \longleftrightarrow df$$

under $\underset{\text{v. fields}}{\mathfrak{X}(M)} \longleftrightarrow \underset{\text{1-forms}}{\Omega^1(M)}$

Poisson bracket: $f, g \in C^\infty(M)$

$$\{f, g\} := \Pi(df, dg)$$

$$= \omega(\xi_f, \xi_g) = \xi_g f = -\xi_f g$$

Properties.

$\{,\}$ is Lie bracket

$\forall f$ $g \mapsto \{f, g\}$ is a derivation:

$$\{f, gh\} = \{f, g\} \cdot h + g \cdot \{f, h\}$$

(Leibniz).

proof. Leibniz follows from $\{f, g\} = -\xi_f g$

Bilinear antisymmetric follows from $\{f, g\} = \omega(\xi_f, \xi_g)$.

Jacobi follows from $d\omega = 0$:

Let $C := \{\{f, g\}, h\}$ + its cyclic permutations

Recall. $(d\omega)(\xi_f, \xi_g, \xi_h) = \xi_f\, \omega(\xi_g, \xi_h)$ + its cyclic permutations

$- \Big(\omega([\xi_f, \xi_g], \xi_h)$ + its cyclic permutations$\Big)$

$$\xi_f\, \omega(\xi_g, \xi_h) = \xi_f \{g, h\} = \{\{g,h\}, f\}$$

this + its cyclic permutations = C.

$$\omega\Big([\xi_f, \xi_g], \xi_h\Big) = -dh\Big([\xi_f, \xi_g]\Big) = -[\xi_f, \xi_g]h$$

$$-\xi_f \xi_g h + \xi_g \xi_f h$$

$$= - \{\{h,g\}, F\} + \{\{h,f\}, g\}$$

$$= \{\{g,h\}, f\} + \{\{h,f\}, g\}$$

this + its cyclic permutations = $2C$.

So $\quad 0 = d\omega(\xi_f, \xi_g, \xi_h) = C - 2C = -C$

so $\quad C = 0$.

Lemma

$$C^\infty(M) \longrightarrow \mathfrak{X}(M)$$

$$f \longmapsto \xi_f$$

is a anti Lie algebra homomorphism.

proof. We need to show: $\forall f, g \in C^\infty(M)$

$$\xi_{\{f,g\}} = -[\xi_f, \xi_g]$$

in $\mathfrak{X}(M)$

Indeed, $\forall h \in C^\infty(M)$

$$
\begin{aligned}
-[\xi_f, \xi_g]\, h &= -\xi_f\, \xi_g\, h + \xi_g\, \xi_f\, h \\
&= -\{\{h, g\}, f\} + \{\{h, f\}, g\} \\
&= \{\{g, h\}, f\} + \{\{h, f\}, g\} \\
&\underset{\text{Jacobi}}{=} -\{\{f, g\}, h\} = \xi_{\{f,g\}}\, h
\end{aligned}
$$

<u>Lemma</u> Let $G \circlearrowright (M, \omega) \xrightarrow{\ \mu\ } \mathfrak{g}^*$

be a Hamiltonian G-mfld.

(with μ an (equivariant) momentum map).

Then

$$
\begin{aligned}
\mathfrak{g} &\longrightarrow C^\infty(M) \\
\xi &\longmapsto \mu^\xi := \langle \underset{\substack{\uparrow \\ \mathfrak{g}^*}}{\mu(\cdot)}, \underset{\substack{\uparrow \\ \mathfrak{g}}}{\xi} \rangle
\end{aligned}
$$

is a Lie algebra homomorphism.

proof $\mu : M \longrightarrow \mathfrak{g}^*$ is equivariant

$$\Rightarrow \quad \forall \xi \in \mathfrak{g} \quad \forall x \in M \qquad \underbrace{d\mu|_x(\xi_M)}_{(\xi_M \mu)(x)} = \underbrace{\xi_{\mathfrak{g}^*}\big|_{\mu(x)}}_{ad^*(\xi)(\mu(x))}$$

in $T_{\mu(x)}\mathfrak{g}^* \cong \mathfrak{g}^*$

So $\forall \eta \in \mathfrak{g}$

$$\langle \underbrace{(\xi_M \mu)(x)}_{\mathfrak{g}^*}, \underbrace{\eta}_{\mathfrak{g}} \rangle = \langle \underbrace{ad^*(\xi)(\mu(x))}_{\mathfrak{g}^*}, \underbrace{\eta}_{\mathfrak{g}^*} \rangle$$

$$\text{LHS} = (\xi_M \mu^\eta)(x) = \{\mu^\eta, \mu^\xi\}(x)$$

Today's convention for momentum maps:

The v. field corresponding to μ^ξ is ξ_M.

$$\text{RHS} = \langle \mu(x), -ad(\xi)\eta \rangle = \langle \mu(x), -[\xi, \eta] \rangle$$
$$= \mu^{[\eta, \xi]}(x)$$

$$\therefore \quad \{\mu^{\zeta}, \mu^{\xi}\} = \mu^{[\xi,\zeta]}$$

i.e. $\xi \mapsto \mu^{\xi}$ is a Lie homomorphism

Moreover Given $G \circlearrowleft (M,\omega)$

and $\mu : M \longrightarrow \mathfrak{g}^*$

that satisfies Hamilton's equations.

$\boxed{\text{Assuming } G \text{ is connected}}$

μ is equivariant if & only if

the map $\mathfrak{g} \longrightarrow C^\infty(M)$

$\xi \mapsto \mu^{\xi}$

is a Lie homomorphism.

Hamiltonian torus actions.

Let $G \cong (S^1)^k$ be a torus.

Note. $\mu : M \to \mathfrak{g}^* \cong \mathbb{R}^k$ is equivariant iff it's invariant.

(the coadjoint action is trivial).

Thm. Let $G \cong (S^1)^k$.
Let $G \circlearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}^*$

s.t. $\forall \xi \in \mathfrak{g} \quad d\mu^\xi = \iota_{\xi_M} \omega$.

Then μ is G-invariant.

Proof. Need to show: μ is constant on G-orbits.
Recall: $\forall x \in M \quad T_x(G \cdot x) = \{ \xi_M|_x \mid \xi \in \mathfrak{g} \}$.

So it's enough to show:

$$\forall \xi \in \mathfrak{g} \quad \underbrace{\mathcal{L}_{\xi_M} \mu}_{\text{also write as } \xi_M \mu.} = 0$$

i.e. $\forall \eta \in \mathfrak{g}$

$$\mathcal{L}_{\xi_M} \mu^\eta = 0 \quad \text{in} \quad C^\infty(M).$$

we'll first show that $\boxed{\mathcal{L}_{\xi_M} \mu^\eta}$ is constant on orbits.

$\forall \zeta \in \mathfrak{g}$

$$\mathcal{L}_{\zeta_M} \boxed{\mathcal{L}_{\xi_M} \mu^\eta} = \mathcal{L}_{\zeta_M} \iota_{\xi_M} \boxed{d\mu^\eta} = \mathcal{L}_{\zeta_M} \iota_{\xi_M} \boxed{\iota_{\eta_M} \omega}$$

$$= \mathcal{L}_{\zeta_M} \omega(\eta_M, \xi_M)$$

$$\overset{\text{Leibniz}}{=} \underbrace{(\mathcal{L}_{\zeta_M} \omega)}_{\substack{= 0 \\ \text{because } \omega \text{ is} \\ G\text{-invt.}}} (\eta_M, \xi_M) + \omega(\overbrace{[\zeta_M, \eta_M]}^{0 \text{ because } G \text{ is abelian}}, \xi_M) + \omega(\eta_M, \underbrace{[\zeta_M, \xi_M]}_{0})$$

$$= 0.$$

$\therefore$ $\mathcal{L}_{\xi_M} \mu^\eta$ is constant along orbits on each orbit: at a point where μ^3 is maximal, $d\mu^3 = 0$, so $\mathcal{L}_{\xi_M} \mu^3 = 0$.

So this constant is zero.

$$\therefore \quad \mathcal{L}_{\xi_M} \mu^3 \equiv 0 .$$ ✓