26/3/2021

Hamilton equation on cotangent bundles
– two examples.

T^*N Notation: $x \in N$, $\gamma_x \in T_x^*N$.

$\pi : T^*N \to N$

tautological 1-form: $\underbrace{\alpha_{taut}\Big|_{\gamma_x}}$ $\big(\xi \big) = \gamma_x \big(\underbrace{\pi_* \xi}_{\text{in } T_x N} \big)$, ξ in $T_{\gamma_x} T^*N$

Canonical 2-form: $\omega_{can} = - d\alpha_{taut}$

in adapted coordinates: $\alpha_{taut} = \sum p_j dq_j$, $\omega_{can} = \sum dq_j \wedge dp_j$

First example

$G \circlearrowright N$. generating vector fields: ξ_N for $\xi \in \mathfrak{g}$

its cotangent lift: $G \circlearrowright T^*N$.

generating vector fields: ξ_{T^*N} for $\xi \in \mathfrak{g}$

ξ_{T^*N} and ξ_N are π-related

$\exists$ Momentum map: $\mu : T^*N \to \mathfrak{g}^*$

$\forall \xi \in \mathfrak{g}$ $\mu^\xi : T^*N \to \mathbb{R}$

$\gamma_x \longmapsto \gamma_x \big(\xi_N \big|_x \big)$

Second example $\quad f \in C^\infty(N)$

pullback: $\quad f \circ \pi \in C^\infty(T^*N)$

Its Hamiltonian flow is (up to sign)

$$\Psi_t : \gamma_x \longmapsto \gamma_x + t\, df\big|_x \quad \text{for } t \in \mathbb{R}.$$

The time one map take the zero section O_N to graph df

In adapted coordinates. $\quad \omega_{can} = \Sigma\, dq_j \wedge dp_j$

(Ψ_t) is generated by the vector field

$$X = \sum_j \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j}$$

$$\iota_X \omega_{can} = - \Sigma \frac{\partial f}{\partial q_j}\, dq_j$$

$$= -d(\pi^* f)$$

Local ("baby") version of Arnold conjecture

(proved by Alan Weinstein)

(M, ω) compact symplectic mfld. $\neq \emptyset$ points.

$H^1(M) = 0$.

Then $\exists$ C^1-neighbourhood Ω of Id in $Symp(M, \omega)$

s.t. $\forall \varphi : M \to M$ in Ω, $\#\{x \mid \varphi(x) = x\} \geq$ Crit M.

Also, $\Omega \subseteq Ham(M, \omega)$.

Moreover, $\forall \varphi \in \Omega$ can be connected to Id by a Hamiltonian isotopy $(\varphi_t)_{t \in [0,1]}$ $\varphi_0 = Id$, $\varphi_1 = \varphi$ with $\varphi_t \in \Omega$ $\forall t$.

<u>Note</u>. Because $H^1(M) = 0$, every smooth family $(\varphi_t)_{t \in [0,1]}$ of symplectomorphisms is a Hamiltonian isotopy.

<u>Indeed</u>. write $\frac{d}{dt} \varphi_t = X_t \circ \varphi_t$.

(φ_t) is a Hamiltonian isotopy iff $\iota_{X_t} \omega$ is exact $\forall t$.

Indeed: If all $\iota_{X_t} \omega$ are exact, we can take

$$H_t(x) = -\int_{x_0}^{x} \iota_{X_t} \omega$$

$$\begin{tikzcd}
 & \Delta \subset \mathcal{O}_{\Delta} \arrow[d, "W"] \subset_{\text{open}} \bar{M} \times M \ , \ -\omega \oplus \omega \\
M \arrow[ur, "x \mapsto (x,x)"] \arrow[ur, "x \mapsto (x, \psi(x))"'] \arrow[dr, "x \mapsto \beta_x"] \arrow[dr, "x \mapsto 0_x"'] & \\
 & 0_M \subset \mathcal{O}_0 \subset_{\text{open}} T^*M \arrow[d, "\pi"] \\
 & M
\end{tikzcd}$$

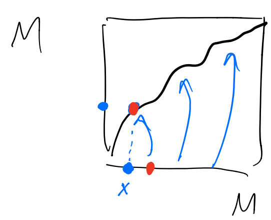
Given

$i : M \to \bar{M} \times M$

$\exists$ diffeo $\psi : M \to M$ s.t. image i = graph ψ

iff $\text{proj}_1 \circ i$ and $\text{proj}_2 \circ i : M \to M$ are diffeo

If so, take $\psi = (\text{proj}_2 \circ i) \circ (\text{proj}_1 \circ i)^{-1}$



Given

$i : M \longrightarrow T^*M$

$\exists \beta \in \Omega^1(M)$ s.t. image i = graph β

iff $\pi \circ i : M \longrightarrow M$ is a diffeomorphism.

If so, take $\beta = i \circ (\pi \circ i)^{-1}$.

It follows that

$$\Omega_1 := \{ i : M \to \bar{M} \times M \mid \text{image } i = \text{graph } \psi \text{ for some diffeo } \psi \} \text{ is } C^1\text{-open}$$

in $\{ M \longrightarrow \bar{M} \times M \}$

and

$$\Omega_2 := \{ i : M \to T^*M \mid \text{image } i = \text{graph } \beta \text{ for some 1-form } \beta \} \text{ is } C^1\text{-open}$$

in $\{ M \to T^*M \}$.

Consider.

$$\Big\{ \psi : M \to M \;\Big|\; \psi \text{ is a diffeomorphism}, \quad \text{graph } \psi \subset \mathcal{O}_\Delta, \quad W(\text{graph } \psi) = \text{graph } \beta \text{ for some } \beta \in \Omega^1(M), \quad \text{graph}(t\beta) \subset \mathcal{O}_0 \;\; \forall t \in [0,1], \quad \forall t \in [0,1] \;\; W^{-1}(\text{graph } t\beta) = \text{graph } \psi_t \text{ for some diffeo } \psi_t \Big\}$$

This ↗ is C^1-open in $\{\text{smooth } M \to M\}$.

For such ψ, we obtain $\{\psi_t\}_{t \in [0,1]}$ s.t.

$$W(\text{graph } \psi_t) = \text{graph}(t\beta).$$

If ψ was a symplectomorphism

Then β is a closed 1-form

so $t\beta$ is a closed 1-form

So ψ_t is a symplectomorphism.