31/3/2021.

Paths in the Hamiltonian group.

Fix a symplectic mfld (M,ω).

Let $\psi_t : M \to M$, for $t \in [0,1]$, be a smooth family of symplectomorphisms.

Assume $\forall t$ $\psi_t \in \mathrm{Ham}(M,\omega)$.

Theorem: $(\psi_t)_{t\in[0,1]}$ is a Hamiltonian isotopy.

proof (Banyaga)

write
$$\frac{d}{dt}\psi_t = X_t \circ \psi_t .$$

$(\psi_t)_{t\in[0,1]}$ is a Hamiltonian isotopy iff $\iota_{X_t}\omega$ is exact for all $t\in[0,1]$.

(Indeed, if so, take $H_t(x) := -\int_{x_0}^{x} \iota_{X_t}\omega$).

This holds iff $\forall t$ $[\iota_{X_t}\omega] = 0$ in $H^1(M,\mathbb{R})$

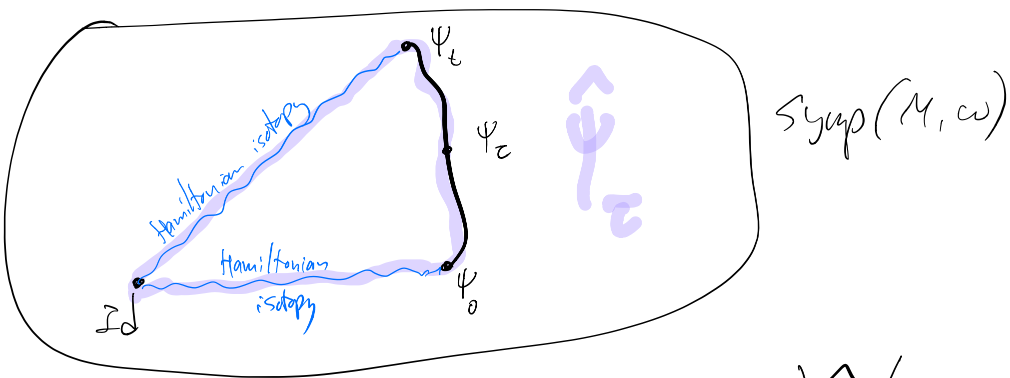
iff $\forall t$
$$\int_0^t \left[\iota_{X_{\cancel{t}\tau}}\omega\right] d\cancel{t}\tau = 0 .$$

$\left([0,1] \longmapsto H^1(M,\mathbb{R})\right)$

$$\int_0^t [\iota_{X_\tau}\omega]\, d\tau = \left[\int_0^t \iota_{X_\tau}\omega \, d\tau\right] \text{ in } H^1.$$

The flux of the symplectic isotopy $(\psi_\tau)_{0\le\tau\le t}$

- it does not change under reparametrization.



$\forall t$ $\exists$ a loop of symplectomorphisms $\{\hat{\psi}_\tau\}_{\tau\in[0,1]}$

obtained by concatenating $\{\psi_\tau\}_{\tau\in[0,t]}$

with Hamiltonian isotopies from Id to ψ_0
& from ψ_t to Id

with the same flux as $\{\psi_\tau\}_{\tau\in[0,t]}$.

We claim: The flux of a (closed) loop of symplectos
is a class in $H^1(M,\mathbb{R})$
whose evaluation on elements of $H_1(M,\mathbb{Z})$
take values in $\langle \omega, H_2(M,\mathbb{Z})\rangle$

$\{$such classes in $H^1(M,\mathbb{R})\}$ is countable.

So
$$t \longmapsto \mathrm{Flux}\left((\psi_\tau)_{\tau\in[0,t]}\right) = \int_0^t \left[\iota_{X_\tau}\omega\right] d\tau$$
is a continuous map $[0,1] \longrightarrow H^1(M,\mathbb{R})$ that takes values in a countable set. So it's constant.

For $t=0$ it's 0.
$\therefore$ it's the zero map.

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proof of the claim.

Let $(\psi_t)_{t\in[0,1]}$ be a closed loop of symplectomorphisms.
$$\psi_0 = \psi_1 = Id.$$

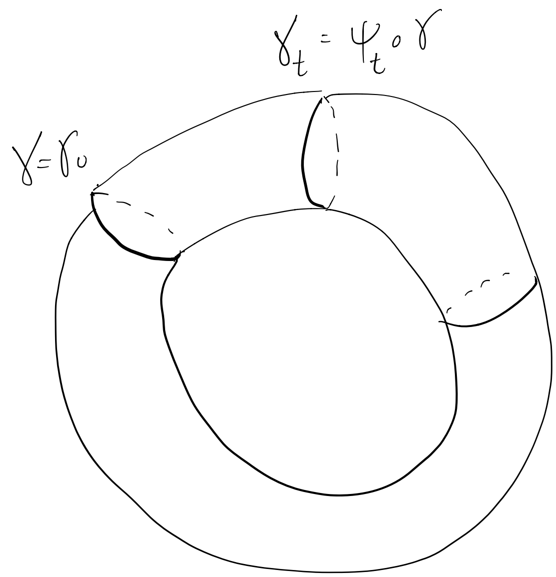
$\frac{d}{dt}\psi_t = X_t \circ \psi_t$ $\qquad$ $\alpha_t := \iota_{X_t}\omega$ are closed.

$\gamma : [0,1] \longrightarrow M$ a loop in M.
$\quad\; s \in [0,1]$ $\qquad \gamma(0)=\gamma(1)$.

we claim that $\left\langle \int_0^1 \alpha_t \, dt, \gamma \right\rangle \in \langle \omega, H_2(M,\mathbb{Z})\rangle$

"Sweep" γ: $\quad \tilde{\gamma} : [0,1]\times[0,1] \longrightarrow M$
$\qquad\qquad (t,s) \longmapsto \gamma_t(s)$

where $\gamma_t = \psi_t \circ \gamma$



Because $\gamma(0) = \gamma(1)$ and $\psi_0 = \psi_1$, $\tilde{\gamma}$ represents a class $[\tilde{\gamma}] \in H_2(M;\mathbb{Z})$, and

$$\int_{[0,1]\times[0,1]} \tilde{\gamma}^*\omega = \langle \omega, [\tilde{\gamma}] \rangle \in \langle \omega, H_2(M,\mathbb{Z}) \rangle$$

claim: this $= \int_\gamma \underbrace{\left(\int_0^1 \alpha_t \, dt \right)}_{\text{closed 1-form}}$

indeed:

$$\langle \omega, [\tilde{\gamma}] \rangle = \int_{[0,1]\times[0,1]} \tilde{\gamma}^*\omega = \int_{[0,1]\times[0,1]} \omega\Big(\underbrace{\tilde{\gamma}_* \tfrac{\partial}{\partial t}}_{X_t|_{\tilde{\gamma}(t,s)}} , \underbrace{\tilde{\gamma}_* \tfrac{\partial}{\partial s}}_{\frac{d}{ds}\gamma_t(s)} \Big) \, dt \wedge ds$$

$$\omega(X_t, \bullet) = \alpha_t$$

$$= \int_{t \in [0,1]} \int_{\gamma_t} \alpha_t \, dt$$

$$= \int_0^1 \left(\int_\gamma \alpha_t \right) dt = \int_\gamma \left(\int_0^1 \alpha_t \, dt \right)$$

γ_t is a closed curve, homotopic to $\gamma_0 = \gamma$

α_t is a closed 1-form

Quadratic Hamiltonians

(V, ω) sympl. vector space.

$$G = Sp(V) = \left\{ A : V \to V \quad \text{s.t.} \quad \forall u, v \quad \omega(Au, Av) = \omega(u, v) \right\}$$

$$\mathfrak{g} = \mathfrak{sp}(V) = \left\{ \alpha : V \to V \quad \text{s.t.} \quad \forall u, v \quad \omega(\alpha u, v) + \omega(u, \alpha v) = 0 \right\}.$$

Claim. $\boxed{1}$ The linear action $G \circlearrowright V$ is Hamiltonian with quadratic momentum map

$$\pm \quad \mu : V \longrightarrow \mathfrak{g}^*$$

given by

$$\mu^\alpha(u) = \frac{1}{2}\omega(\alpha u, u)$$

$\forall \alpha \in \mathfrak{g}$

$\forall u \in V$

$\boxed{2}$ The map $\mu^\# : \mathfrak{g} \longrightarrow \left\{ \begin{matrix} \text{quadratic} \\ \text{functions} \\ \text{on } V \end{matrix} \right\}$

$$\alpha \longmapsto \mu^\alpha$$

is a Lie algebra isomorphism wrt the Poisson bracket on the target.

Recall.

$$\left\{ \begin{array}{c} \text{bilinear forms} \\ B: V\times V \to \mathbb{R} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{linear maps} \\ \alpha: V \to V \end{array} \right\}$$

via $\quad B(u,v) = \omega(\alpha u, v)$

Note $\quad B(u,v) = B(v,u) \quad$ iff $\quad \omega(\alpha u, v) + \omega(u, \alpha v) = 0$

So:

$$\left\{ \begin{array}{c} \text{symmetric} \\ \text{bilinear forms} \end{array} \right\} \longleftrightarrow sp(V,\omega) = \mathfrak{g}$$

$$\omega(\alpha u, v) \longleftrightarrow \alpha$$

Also recall:

$$\left\{ \begin{array}{c} \text{symmetric} \\ \text{bilinear forms} \\ V\times V \to \mathbb{R} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{quadratic} \\ \text{forms} \\ V \to \mathbb{R} \end{array} \right\}$$

$$B(u,v) \longleftrightarrow \frac{1}{2} B(u,u)$$

$\therefore \quad sp(V,\omega) \longleftrightarrow \{\text{quadratic forms}\}$

$$\alpha \longleftrightarrow \frac{1}{2}\omega(\alpha u, u)$$

It remain to prove that the map $\mu: V \to \mathfrak{g}^*$ given by $\mu^\alpha(u) = \frac{1}{2}\omega(\alpha u, u)$ is an (equivariant) momentum map for $G \circlearrowright V$.

Note.

$$\alpha \in \mathfrak{g}^{\subseteq\{V\to V\}} \rightsquigarrow \alpha_V \in \mathfrak{X}(V)$$

$G \subseteq \{V \to V\}$

is given by $\quad \underbrace{\alpha_V\Big|_u}_{T_u V} = \underbrace{\alpha u}_{V}$

$T_u V \xleftrightarrow[\text{identify}]{} V$

$$d\mu^\alpha\Big|_u (v) = \text{derivative of } u \mapsto \tfrac{1}{2}\omega(\alpha u, u) \text{ along } v$$

$$= \frac{1}{2}\Big(\underbrace{\omega(\alpha v, u)}_{= -\omega(v, \alpha u)} + \omega(\alpha u, v)\Big) \quad \text{because } \alpha \in sp(V, \omega)$$

$$= \omega(\alpha u, v)$$

$$= \left(\iota_{\alpha_V} \omega\right)(v)$$

$$\therefore \quad d\mu^\alpha\Big|_u = \iota_{\alpha_V}\omega$$

Equivariance: Let $A \in G$ and $u \in V$.

$$\mu(Au) \overset{?}{=} Ad^*(A)(\mu(u)) \quad \text{in } \mathfrak{g}^*$$

$$\forall \alpha \in \mathfrak{g} \quad \langle \underset{\mathfrak{g}^*}{\underset{\cap}{LHS}}, \underset{\mathfrak{g}}{\underset{\cap}{\alpha}} \rangle = \omega(\alpha A u, A u) \underset{A \in Sp(V, \omega)}{\uparrow =} \omega(A^{-1} \alpha A u, u)$$

$$= \langle \mu(u), A^{-1} \alpha A \rangle = \langle \mu(u), Ad(A^{-1})(\alpha) \rangle$$

$$= \langle Ad^*(A) \mu(u), \alpha \rangle$$

$$= \langle RHS, \alpha \rangle .$$

Symplectic slice representation.

$$G \circlearrowright (M, \omega) \xrightarrow[\mu]{} \mathfrak{g}^*$$

$x \in M$. $H := \mathrm{Stab}(x)$.

$H \circlearrowright (T_x M, \omega_x)$ isotropy repr. at x.

preserves $T_x(G\cdot x)$, $T_x(G\cdot x)^{\omega_x}$.

$$\Rightarrow \quad H \circlearrowright V := T_x(G\cdot x)^{\omega} \big/ T_x(G\cdot x) \cap T_x(G\cdot x)^{\omega}$$

+ sympl. str ω_V induced from ω_x.

Def.

$H \circlearrowright (V, \omega_V)$ is the symplectic slice representation at x.

From earlier in the semester:
(29/1/2021)

$$\begin{tikzcd} G\cdot x \arrow[r, hook, "i"] \arrow[dr, "\mu|_{G\cdot x}"'] & (M,\omega) \arrow[d, "\mu"] \\ & \mathfrak{g}^* \end{tikzcd}$$

$\alpha := \mu(x) \in \mathfrak{g}^*$

$$\underset{\text{transitive}}{G \circlearrowright} (G\cdot x,\ i^*\omega) \xrightarrow[\mu|_{G\cdot x}]{} \mathfrak{g}^*$$

$$i^*\omega = \mu^*\ \omega_{KKS}$$

on $Ad^*(G)\circ\alpha$

Thm. A nbhd of $G\cdot x$ is determined up to equivariant symplectomorphism preserving μ

by the sympl. slice repr. $H \circlearrowright (V, \omega_V)$ and by $\mu(x) = \alpha$.

idea: use Weinstein's tubular nbhd thm...

Local normal form

$$\left(\begin{matrix}\text{nbhd of } G\cdot x \\ \text{in } G \circlearrowright (M,\omega) \xrightarrow[\mu]{} \mathfrak{g}^*\end{matrix}\right) \cong \left(\begin{matrix}\text{nbhd of 0-section in} \\ \text{Hamiltonian } G \text{ model}\end{matrix}\right).$$

$\alpha = \mu(x)$ $\qquad H = \mathrm{Stab}(x) \subset G$

$$G_\alpha := \mathrm{Stab}(\alpha) \text{ under } Ad^*: G \circlearrowright \mathfrak{g}^*.$$

$$H \subset G_\alpha \subset G.$$

$H \circlearrowright (V, \omega_V) \xrightarrow{\mu_V} \mathfrak{h}^*$ $\qquad$ sympl. slice repr.

choose Ad^*-invariant inner product on $\mathfrak{g}^*$

$$\mathfrak{g}^* \cong \left(\mathfrak{g}/\mathfrak{g}_\alpha\right)^* \oplus \left(\mathfrak{g}_\alpha/\mathfrak{h}\right)^* \oplus \mathfrak{h}^*$$

The Model: $\quad Y = G \times_H \left(\left(\mathfrak{g}_\alpha/\mathfrak{h}\right)^* \times V\right) \xrightarrow{\mu_Y} \mathfrak{g}^*$

$$\mu_Y([g,\beta,z]) = Ad^*(g)(\alpha + \beta + \mu_V(z))$$

ω_Y closed G-invt 2-form on Y

Non deg along 0-section.

If $G_x = G$: Nondeg everywhere.

Guillemin-Sternberg - Marle

Momentum maps & Morse theory

compact $G \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$

$\xi \in \mathfrak{g}$ $\qquad \mu^\xi : M \longrightarrow \mathbb{R}$

is a Morse-Bott function.

Moreover:

$\mathrm{Crit}\,\mu^\xi := \{ x \mid d\mu^\xi|_x = 0 \}$

is a locally finite disjoint union of symplectic submflds C of M.

Along each C, the Hessian $\mathrm{Hess}\,\mu^\xi$ is nondegenerate on the fibres of the normal bundle $\nu_M C$ and has even index.

Recall $f : M \to \mathbb{R}$. (smooth) $\qquad x \in \mathrm{Crit} f$.

$\mathrm{Hess}_x f : T_x M \times T_x M \longrightarrow \mathbb{R}$ symmetric bilinear

in local coordinates $t_1, \dots, t_n$: repr. by $\left[\frac{\partial^2 f}{\partial t_i \partial t_j} \right]$

Consequence. $\boxed{\text{If } M \text{ is connected}}$

- level sets of μ^ξ are connected
- μ^ξ has no local min/max except global min/max which is attained on a connected set.