7/4/2021

Tori in Lie groups

source: Adams "Lectures on Lie groups".

G compact connected Lie group

$\forall\ \xi \in \mathfrak{g} = \mathrm{Lie}(G)$ $\quad \overline{\exp(\mathbb{R}\xi)} \subset G$ is a torus.

Conversely:

$\underset{\text{torus}}{\forall\ T \subset G}$ $\quad \exists\ \xi \in \underset{\mathrm{Lie}(T)}{\overset{\shortparallel}{\mathfrak{t}}}$ s.t. $T = \overline{\exp(\mathbb{R}\xi)}$

Let $G \circlearrowright (M, \omega) \xrightarrow[\mu]{} \mathfrak{g}^*$ be a Hamiltonian G-mfld.

Let $\xi \in \mathfrak{g}$. Then

$$\mathrm{Crit}\,\mu^\xi = \{x \in M \mid \xi_M|_x = 0\} = M^T$$

where $T = \overline{\exp(\mathbb{R}\xi)}$

$\therefore$ $\mathrm{Crit}\,\mu^\xi$ is a locally finite disjoint union of symplectic submflds.

Maximal tori

a <u>torus</u> in G is a Lie subgroup that is $\cong (S^1)^k$ for some k.

a <u>maximal torus</u> in a Lie group G is a torus $T \subset G$ s.t. $\forall$ torus $\hat{T} \subset G$ if $T \subseteq \hat{T}$ then $T = \hat{T}$.

Assume G is compact & connected.

- such T exist (easy)
- any element of G is contained in such a T. (Nontrivial, uses Lefschetz fixed point theorem)
- any two maximal tori are conjugate: $T_2 = g T_1 g^{-1}$.

equivalent to: $\exp : \mathfrak{g} \to G$ is onto.

<u>Example</u> $G = U(n)$

$$T = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \;\middle|\; \begin{matrix} \forall j \\ |a_j| = 1 \end{matrix} \right\} \cong (S^1)^n$$

<u>Note</u> Not every abelian subgroup is contained in a torus.

e.g. $$\left\{ \begin{pmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix} \right\} \subseteq SO(n)$$

The standard maximal torus of $SO(n)$ is

n even: $\left\{ \begin{pmatrix} \Box & & & \\ & \Box & & \\ & & \ddots & \\ & & & \Box \end{pmatrix} \right\}$ each block: $\left\{ \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \,\middle|\, c^2+s^2=1 \right\}$

$Ad^* : G \circlearrowright \mathfrak{g}^*$

For generic $\beta \in \mathfrak{g}^*$, $\quad$ $\mathrm{stab}(\beta)$ is a maximal torus in G.

Means: $\{$such $\beta\} \subset \mathfrak{g}^*$ is residual: "big" in the sense of Baire category thm i.e. contained in a countable intersection of open dense sets.

In fact, $\{$such $\beta\}$ is itself open & dense.
It is the pricipal orbit type stratum in $\mathfrak{g}^*$
Its complement is a finite union of proper submflds of $\mathfrak{g}^*$.

Example $\qquad G = U(n)$ $\qquad \mathfrak{g} = \mathfrak{u}(n) = \mathrm{Lie}(G)$

$= \left\{ \begin{array}{c} \text{anti Hermitian} \\ \text{matrices} \end{array} \right\}$

$Ad^* : G \circlearrowright \mathfrak{g}^*$ $\qquad \overline{\alpha}^T = -\alpha.$

identify $\quad \mathfrak{g}^* \longleftrightarrow \{\text{Hermitian matrices}\}$

β s.t. $\overline{\beta}^T = \beta$

$\qquad \psi$

$\beta : \alpha \longmapsto \underbrace{\mathrm{Trace}(i\alpha\beta)}_{\text{real}}$

$Ad^* \longleftrightarrow$ conjugation

$\beta \in \mathfrak{g}^*$ $\qquad$ Hermitian

$$\mathrm{stab}(\beta) = \{A \in G \mid A\beta A^{-1} = \beta\}.$$

is a torus iff the eigenvalues of β are distinct.

If so, up to change of basis

($\Leftrightarrow$ conjugation by an element of $G = U(n)$)

$$\beta = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \qquad d_1, \dots, d_n \in \mathbb{R} \quad \text{distinct}$$

$\mathrm{Stab}(\beta) = T$ $\qquad$ the standard maximal torus.

Relation with symplectic geometry

G compact connected

$\beta \in \mathfrak{g}^*$ s.t. $T := \mathrm{Stab}(\beta) \subset G$ is a maximal torus

$M :=$ the coadjoint orbit through β.

$$G \circlearrowright \underset{\underset{G/T}{\text{\rotatebox{90}{$\cong$}}}}{(M, \omega)} \xrightarrow{\mu} \mathfrak{g}^*$$

Let $\hat{T} < G$ be any other torus.

Let $\xi \in \hat{\mathfrak{t}}$ be s.t. $\hat{T} = \overline{\exp(\mathbb{R}\xi)}$

$$M^{\hat{T}} = \underbrace{\mathrm{Crit}\, \mu^{\xi}}_{\# \geq \mathrm{Crit}\, M \geq 2} \neq \phi$$

Let $x \in M^{\hat{T}}$. write $x = Ad^*(a)(\beta)$.

identifying $M = G/T$: $x = aT$.

$\forall g \in \hat{T}$ $g \cdot x = x$

so $gaT = aT$

so $a^{-1}gaT = T$

$\therefore$ $a^{-1}\hat{T}a \subset T$

i.e. $\forall$ torus $\hat{T}$ in G is conjugate to a subtorus of T.

Now Take arbitrary $g \in G$.

a path from 1 to g

$\rightsquigarrow$ Hamiltonian isotopy of the coadjoint orbit (M, ω)

$x \in M$ is fixed by g

$\Longleftrightarrow$ identifying $M \longleftrightarrow G/T$,
$x \longleftrightarrow aT$

$$g a T = a T$$

$\Longleftrightarrow$ $a^{-1} g a \in T$.

Thus, "every element of G is contained in a conjugate of T"
is an illustration of Arnold's conjecture:
the Hamiltonian symplectomorphism $g \curvearrowright M$ has a fixed point.

Local normal form for Hamiltonian torus actions

$$T \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{t}^*$$

$T \cong (S^1)^k$, $\mathfrak{t}^* \cong \mathbb{R}^k$

T-orbits are isotropic:

$\forall\ \xi, \eta \in \mathfrak{t} = \mathrm{Lie}(T)$

$$\omega(\xi_M, \eta_M) \underset{\pm}{=} (d\mu^\xi)(\eta_M) = \eta_M \mu^\xi = 0$$

μ is T-invariant

So $\forall x \in M$ $\quad \underbrace{T_x}_{\text{its tangent}} \underbrace{(T \cdot x)}_{\text{the orbit}} \subset T_x M$

$$T_x(T\cdot x)^\omega \supseteq T_x(T\cdot x)$$

So

$$H := \mathrm{stab}(x) \circlearrowright \underset{\text{symplectic slice}}{V} \xrightarrow{\mu_V} \mathfrak{h}^*$$

with $\quad V = \dfrac{T_x(T\cdot x)^\omega}{T_x(T\cdot x)}$

Local normal form: nbhd of x in $M \longleftrightarrow$ nbhd of 0 section in $T \times_H V \times \mathfrak{h}^0$

$$x \longleftrightarrow [1, 0, 0]$$

$$\mu([g, z, \nu]) = \underbrace{\mu_V(z)}_{\in \mathfrak{h}^*} + \underbrace{\nu}_{\mathfrak{h}^0} \quad \in \mathfrak{h}^* \times \mathfrak{h}^0 \cong \mathfrak{t}^*$$

Linear symplectic torus actions.

$$\begin{tikzcd}
T \arrow[d, hook, "i"'] & \circlearrowright & (V, \omega_V) \arrow[rr, "\mu"', "\text{quadratic}"] \arrow[d, phantom, "\cong" rotate=90] & & \mathfrak{t}^* \\
(S^1)^n & \circlearrowright & (\mathbb{C}^n, \omega_{std}) \arrow[rr] \arrow[urr, "\mu_T"] & & \mathbb{R}^n \arrow[u, "i^*"']
\end{tikzcd}$$

i^* linear projection

$$(z_1, \dots, z_n) \longmapsto \left(\frac{|z_1|^2}{2}, \dots, \frac{|z_n|^2}{2}\right)$$

$i : T \hookrightarrow (S^1)^n$

$i_* : \mathfrak{t} \hookrightarrow \mathbb{R}^n$

$i^* : (\mathbb{R}^n)^* \longrightarrow \mathfrak{t}^*$

$$\mu_T(z_1, \dots, z_n) = \sum_{j=1}^{n} \frac{|z_j|^2}{2} \alpha_j$$

$\alpha_1, \dots, \alpha_n$ are the weights for the torus action on V.

$i : T \hookrightarrow (S^1)^n$

each component :

$$\begin{tikzcd} T \arrow[r] & S^1 \\ \mathfrak{t} \arrow[u, "\exp"] \arrow[r, "\alpha_j"'] & \mathbb{R} \arrow[u, "\exp"'] \end{tikzcd}$$

homomorphism

$$\mathrm{Hom}(T, S^1) \cong \underbrace{\mathfrak{t}^*_{\mathbb{Z}}}_{\text{The lattice of weights.}} \subset \mathfrak{t}^*$$

e.g. $T = (S^1)^k$.

$m_1, \dots, m_k \in \mathbb{Z}$

$$\begin{tikzcd} (S^1)^k \arrow[r, "(a_1, \dots, a_k) \mapsto a_1^{m_1} \cdots a_k^{m_k}"'] & S^1 \\ \mathbb{R}^k \arrow[u, "\exp"] \arrow[r, "(x_1, \dots, x_k) \mapsto \sum m_j x_n"'] & \mathbb{R} \arrow[u, "\exp"'] \end{tikzcd}$$

so $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$

$$\begin{matrix} \mathfrak{t}^*_{\mathbb{Z}} & \subset & \mathfrak{t}^* \\ \wr\!\parallel & & \wr\!\parallel \\ \mathbb{Z}^k & \subset & \mathbb{R}^k \end{matrix}$$

Note. for $T \circlearrowright (V, \omega_V) \xrightarrow{\mu_T} \mathfrak{t}^*$

$$\text{image}\, \mu_T = \sum_{j=1}^{n} \mathbb{R}_{\geq 0}\, \alpha_j$$

where $\alpha_1, \dots, \alpha_n$ are the weights for the T-action.

So image μ_T is a convex polyhedral cone.

<u>Condevaux – Dazord – Molino</u> approach to convexity properties of momentum maps.

Def. a continuous map

$$\mu : \underset{\text{connected Hausdorff}}{X} \longrightarrow \mathbb{R}^k$$

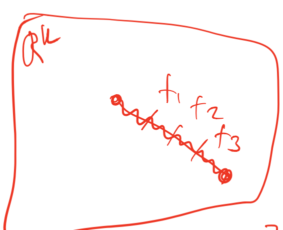
is CDM if every two points in X can be connected by a continuous path $\gamma : [0,1] \longrightarrow X$ s.t.

$$\mu \circ \gamma : [0,1] \longrightarrow \mathbb{R}^k \quad \text{is "monotone straight"}$$

i.e. is a $\boxed{\text{weakly monotone}}$ parametrization of a (possibly degenerate) closed interval.

CDM implies:

- X is connected
- image is convex
- level sets are connected



i.e. $\forall x_0, x_1 \in X$

$\exists \gamma : [0,1] \longrightarrow X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$

$\forall \; 0 \leq t_1 \leq t_2 \leq t_3 \leq 1$

$$\mu(\gamma(t_2)) \in [\mu(\gamma(t_1)), \mu(\gamma(t_3))]$$

Special case: $\mu : X \hookrightarrow \mathbb{R}^k$ inclusion map of a subset

- μ is CDM iff X is convex
- μ is proper iff X is closed

Condevaux - Dazord - Molino
Hilgert - Neeb - Plank
See: | Bjorndahl - Karshon
Ratiu - Birtea - Ortega.

Lokal-global-prinzip

X connected Hausdorff

$\mu : X \longrightarrow \mathbb{R}^k$ proper

or: proper as a map to a convex subset of $\mathbb{R}^k$

If each point of X has a nbhd U s.t.

$$\mu|_U : U \longrightarrow \mu(U) \subset \mathbb{R}^k$$

is CDM & open

Then $\mu : X \longrightarrow \mu(X)$ is also CDM & open.

special case:

1928 Tietze-Nakajima:

$X \subseteq \mathbb{R}^k$ closed, connected, locally convex

Then X is convex.

$\forall$ pt $\exists$ nbhd U s.t. $U \cap X$ is convex

<u>proposition</u> $\mathbb{R}^n_{\geq 0} \xrightarrow[\text{linear projection}]{} \mathbb{R}^k$

$$(s_j) \longmapsto \sum s_j \alpha_j$$

is CDM & is open as a map to its image.

This + local normal form implies: $\forall$ Hamiltonian T-mfld

$$T \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{t}^*$$

$\forall$ pt in M $\exists$ nbhd $\mathcal{U}$ s.t.

$$\mu|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mu(\mathcal{U})$$

is CDM & is open.

<u>Corollary</u>. $\boxed{\text{Convexity package}}$ for Hamiltonian torus actions:

$$T \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{t}^*$$

assume M is connected

M is compact

or: μ is proper

or: μ is proper as a map to some convex subset of $\mathfrak{t}^*$.

Then $\mu : M \to \mathfrak{t}^*$ is CDM
& is open as a map to its image.

<u>Corollary</u> $\mu(M)$ is convex
$\mu^{-1}(\{p\})$ are connected.

(Originally proved by Guillemin–Sternberg & Atiyah
using Morse theory for components of the momentum map.

Nonabelian version: Kirwan.)

Moreover:

$$T \circlearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{t}^*$$

connected & compact

Then $\mu(M) = \mathrm{Conv}\, \mu(M^T)$

Note: $\mu|_{M^T} : M^T \to \mathfrak{t}^*$ is locally constant.

and $\quad |\pi_0(M^T)| < \infty$

So $\quad \big|\mu(M^T)\big| < \infty$

so $\quad \mu(M)$ is a convex polytope.

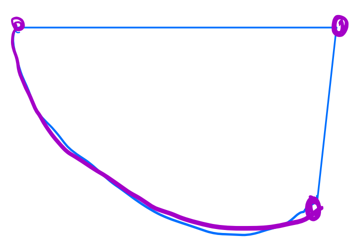
The prove uses the finite dim'l version of Krein-Milman:

∀ compact convex subset of $\mathbb{R}^k$ is the convex hull of its extremal points.

$\alpha \in A \subset \mathbb{R}^k$ compact convex

α is extremal if $\forall\, \alpha_0, \alpha_1 \in A$
if $\exists\; 0 < t < 1$ s.t. $\alpha = (1-t)\alpha_0 + t\alpha_1$
Then $\alpha_0 = \alpha_1 = \alpha$.

e.g.



extremal pts.

It remains to show: $\mu(M^T) \supseteq \{$the extremal pts of $\mu(M)\}$

i.e., if $x \notin M^T$ then $\mu(x)$ is not extremal in $\mu(M)$.