12/4/2021

Today: a bit of classical mechanics.

Noether's principle.

(M, ω) phase space of a physical system

a point in M encodes a state of our system.

positions + velocities/momenta of the particles

The Hamiltonian = energy function

$$H : M \longrightarrow \mathbb{R}$$

generates a v. field $X \in \mathfrak{X}(M)$

whose flow is time evolution.

The flow in canonical coordinates: its trajectories are

$\big(q_j(t), p_j(t)\big)$ s.t.

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Symmetries: $G \circlearrowright (M, \omega) \xrightarrow{\mu} \mathfrak{g}^*$

s.t. H is G-invariant.

Theorem "Conservation of momentum":

The coordinates of μ are preserved under time evolution.

Proof $\mathcal{L}_X \mu^\xi \underset{\pm}{=} \{\mu^\xi, H\} = -\mathcal{L}_{\xi_M} H \underset{H \text{ is } G\text{-invt}}{=} 0 \quad \forall \xi \in \mathfrak{g}$

"Shortcut to Hamiltonian mechanics"

n particles. positions: $q_i \in \mathbb{R}^3$
$i = 1, \ldots, n$

masses: m_i

potential:

$$\mathcal{U} = \mathcal{U}(q_1, \ldots, q_n) : (\mathbb{R}^3)^n \longrightarrow \mathbb{R}$$

Newton's equations: $\underbrace{m_i \ddot{q}_i}_{\text{acceleration}} = -\operatorname{grad} \mathcal{U}$

e.g. gravitation: $$U = -\sum_{\substack{i,j \\ i \neq j}} \frac{m_i m_j}{\| x_i - x_j \|} G$$

$$m_i \ddot{x}_i = \sum_{\substack{j \\ j \neq i}} \frac{m_i m_j (x_j - x_i)}{\| x_j - x_i \|^3} G$$

Momentum coordinates: $p_i = m_i \dot{q}_i \in \mathbb{R}^3$

$\Bigg($ $Q = \mathbb{R}^{3n}$
metric given at each pt by the matrix $\begin{pmatrix} m_1 & & & & & & 0 \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & m_2 & & & \\ & & & & m_2 & & \\ & & & & & m_2 & \\ 0 & & & & & & \ddots \end{pmatrix}$
identifies TQ with T^*Q $\Bigg)$

Energy function:

$$H = \underbrace{\sum_i \frac{1}{2} m_i \|\dot{q}_i\|^2}_{\text{kinetic energy}} + \underbrace{U(\vec{q})}_{\substack{\text{potential}\\ \text{energy}}}$$

$$= \sum \frac{1}{2m_i} \|p_i\|^2 + U(\vec{q})$$

"momentum"

phase space: coordinates $q_i \in \mathbb{R}^3$

& $p_i \in \mathbb{R}^3$

Newton equations $\Longleftrightarrow$ Hamilton's equations:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$$m_i \ddot{q}_i = -\frac{\partial U}{\partial q_i} \qquad \dot{q}_i = \frac{p_i}{m_i}$$

Geometric mechanics.

$\longrightarrow$ Kinematics: configuration space, phase space

$\longrightarrow$ Dynamics: equations of motion = time evolution

configuration of a system: list of positions of its particles

Configuration space:

$$Q = \{\text{all possible configurations}\}$$

modelled by a smooth mfld.

State of a system: List of positions & velocities/momenta of its particles

"velocity phase space" = {states} = $TQ \cong T^*Q$

holonomic system

Legendre transform

Lagragian mechanics

Hamiltonian mechanics

Note Newton's equations were 2nd order ODEs on Q.

$\Longleftrightarrow$ 1st order ODE TQ
i.e. vector field

Holonomic: constraints on velocities only come from constraints on positions.

Nonholonomic example: ball rolling on table without sliding.

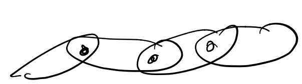
$$Q = \mathbb{R}^2 \times SO(3)$$

Velocity phase space $= E \subsetneq TQ$

The $SO(3)$-velocity determines the $\mathbb{R}^2$-velocity.

Examples of holonomic systems:

	Q	TQ
n noncolliding particles	$(\mathbb{R}^3)^n \setminus$ diagonals	$\left((\mathbb{R}^3)^n \setminus \text{diagonals}\right) \times (\mathbb{R}^3)^n$
planar pendulum	S^1	$S^1 \times \mathbb{R}$
spherical pendulum	S^2	$TS^2 \neq S^2 \times \mathbb{R}^2$
rigid body attached at one point	$SO(3)$	$SO(3) \times \mathbb{R}^3$
rigid body	$SE(3) = SO(3) \ltimes \mathbb{R}^3$	$SE(3) \times \mathbb{R}^6$
Robot with 4 bodies & 3 joints	$Q \subseteq SE(3)^4$	TQ



Lagrangian mechanics . in TQ

motion determined by the Lagrangian

$$\mathcal{L} : TQ \longrightarrow \mathbb{R} .$$

e.g. $\mathcal{L} =$ kinetic energy $-$ potential energy

action of a path $\gamma : [a,b] \longrightarrow Q$

is $$A_\gamma := \int_a^b \mathcal{L}(\gamma, \dot\gamma)\, dt$$

action functional : $\{\gamma\} \longrightarrow \mathbb{R}$ $\quad \gamma \mapsto A_\gamma$

Hamilton's principle of least action :

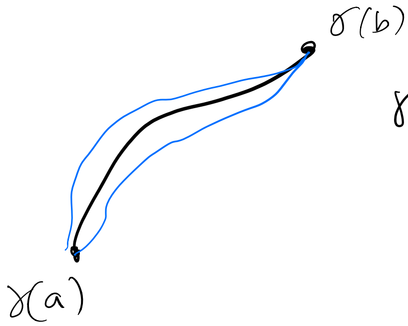
The physical path is stationary for the action functional among variations with fixed endpoints :

$\gamma : [a,b] \longrightarrow Q$

variation : $\quad \gamma_\varepsilon : [a,b] \longrightarrow Q \qquad \varepsilon \in \mathbb{R}$

s.t. $\quad \gamma_\varepsilon(a) = \gamma(a), \quad \gamma_\varepsilon(b) = \gamma(b), \quad \gamma_0 = \gamma$

$\forall \varepsilon$



γ stationary means: $\forall$ such γ_ε

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A_{\gamma_\varepsilon} = 0$$

global

• γ is stationary for A_γ
iff γ satisfies Euler-Lagrange equations

infinitesimal

in adapted coordinated:

for $\mathcal{L} = \mathcal{L}(x, v)$: $\quad \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial v} = \dfrac{\partial \mathcal{L}}{\partial x}$

Legendre transform.

Lagrangian formalism $\rightsquigarrow$ Hamiltonian formalism.

Given $\quad \mathcal{L} = \mathcal{L}(x, v) : TQ \longrightarrow Q$.

Produce $\quad$ • $\Psi : TQ \longrightarrow T^*Q$

If Ψ is invertible: • $H : T^*Q \longrightarrow \mathbb{R}$

Thm $\gamma : [a,b] \longrightarrow Q$
satisfies Euler-Lagrange for $\mathcal{L}$

$\iff$ $\Psi(\gamma,\dot\gamma) : [a,b] \longrightarrow T^*Q$

satisfies Hamilton's equations for H.

Moreover, every solution of Hamilton's equations for H
has this form.

The Legendre transform.

In adapted coordinates: $\Psi(x,v) = (x,p)$

where $\forall j$ $\quad p_j = \dfrac{\partial \mathcal{L}}{\partial v_j}$.

without coordinates:

$\forall x \in Q$ Let $f : \mathcal{L}\big|_{T_xQ} : T_xQ \to \mathbb{R}$

$\forall u \in T_xQ$ $\quad df\big|_u \in T_x^*Q$

Ψ takes $\underset{T_xQ}{\overset{\cap}{u}}$ to this element of T_x^*Q

i.e. Ψ is the "vertical differential" of $\mathcal{L} : TQ \to \mathbb{R}$
"differential along the fibres"

If $\quad \Psi : TQ \to T^*Q \quad$ is invertible

Take $\quad H : T^*Q \longrightarrow \mathbb{R}$ to be

$$H = \langle p, v \rangle - \mathcal{L}(x, v)$$

$T^*Q \qquad TQ$

where $\quad v = v(x, p)$

In our example, $\quad \Psi : TQ \to T^*Q$

comes from the R. metric $\begin{pmatrix} m_1 & & & & \\ & m_1 & & & \\ & & m_1 & & \\ & & & m_2 & \\ & & & & \ddots \end{pmatrix}$

$$\mathcal{L} = \Sigma \tfrac{1}{2} m_i \|v_i\|^2 - U(\vec{q})$$

$$\langle p, v \rangle = \sum_i m_i \|v_i\|^2$$

$$\langle p, v \rangle - \mathcal{L}(x, v) = \Sigma \tfrac{1}{2} m_i \|v_i\|^2 + U(\vec{q})$$

$$= \Sigma \frac{1}{2m_i} \|p_i\|^2 + U(\vec{q})$$

$v = v(x, p)$