MAT157 - Analysis I, 2018–19. Assignment 1.

Read Chapter 1 of Spivak's book, including the exercises. Read slowly the notes "Postulate for the real numbers and some consequences" and "Approximations" on the course website.

Prepare solutions to the following problems for Thursday September 13th; submit them in class on Tuesday September 18th. In proofs, state explicitly every fact that you're using.

(1) Prove the Cauchy-Schwarz inequality: $x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$.
Moreover, prove that it is an equality, $x_1y_1 + x_2y_2 = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$, if and only if $x_1 = x_2 = 0$ or $y_1 = y_2 = 0$ or there exists $\lambda > 0$ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

Hint: apply the inequality $2xy \leq x^2 + y^2$ (why is it always true?), which is an equality if and only if $x = y$ (why?), to $x = \frac{x_i}{\sqrt{x_1^2+x_2^2}}$ and $y = \frac{y_i}{\sqrt{y_1^2+y_2^2}}$, for $i = 1$ and for $i = 2$.

(2) Prove two of the theorems/corollaries in the handout "Postulates for the real numbers and some consequences", assuming only earlier items from that handout.

(3) Let u and v be positive real numbers.
 (a) Prove that $u^2 - v^2 = (u + v)(u - v)$, where $x^2 := x \cdot x$.
 (b) Prove that, if $u > v$, then $u^2 > v^2$.
 (c) Prove that, if $u^2 > v^2$, then $u > v$.
You may use the items from the handout "Postulates for the real numbers and some consequences", and nothing else.

(4) Assume that $|x - 2\pi| < \frac{1}{1000}$ and $|y - e| < \frac{1}{1000}$.
 (a) Prove that the distance between $x + y$ and $2\pi + e$ is $< \frac{1}{100}$.
 (b) Prove that the distance between xy and $2\pi \cdot e$ is $< \frac{1}{50}$.
- You may use the facts that $6 < 2\pi < 7$ and $2 < e < 3$.
- Do not quote from the handout "Approximations", but you can imitate its tricks.
- Disclaimer: whereas each of the constants 2π and e is important on its own, I doubt that the sum $2\pi + e$ or product $2\pi \cdot e$ is ever useful.

(5) The golden ratio $\varphi := \frac{1+\sqrt{5}}{2}$ has some wonderful properties (you can look it up).
 (a) Prove that $1 < \varphi < 2$.
 (b) Find a positive number E such that, if $|x - \varphi| < E$, then $|\frac{1}{x} - \frac{1}{\varphi}| < \frac{1}{100}$.
 Prove that the number E that you found indeed satisfies this condition.

Make sure that you can solve the following problems. Do not hand in your solutions.

- Prove all the theorems and corollaries in the handout "Postulates for the real numbers and some consequences". For each item, you may rely only on earlier items.
- Spivak, Chapter 1: Problems 2 (flawed proof); 3 (ii),(v),(vi) (arithmetic of fractions); 4 (ix), (x), (xi) (sets given by inequalities); 5 (vi) and (vii) (comparing a^2 and a); 9 (iii),(v) (simplify expression with absolute values); 18 (b),(e) (completing the square).