MAT157 – Analysis I, 2018–19. Assignment 4.

Read Chapters 5 and 6 of Spivak's book. Clear solutions to the following problems are due in class on Thursday October 18th.

(1) Suppose that $f(x) \leq g(x) \leq h(x)$ for all x and that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = \ell$. Prove that $\lim_{x\to\infty} g(x) = \ell$.

(2) Give one example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies $\lim\limits_{x\to 2} f(x) = 10$ but that does not satisfy the following condition.

($\star$) There exists $\delta > 0$ such that for all $\epsilon > 0$, if $0 < |x-2| < \delta$, then $|f(x) - 10| < \epsilon$.

Explain.

(3) For each $n \in \mathbb{N}$, let A_n be a finite set of numbers. Assume that for each n and m such that $n \neq m$, the sets A_n and A_m are disjoint (do not have any common elements). Define $f(x) = \dfrac{1}{n}$ if $x \in A_n$; define $f(x) = 0$ if x is not in A_n for any n.
   (a) Prove that $\lim\limits_{x\to 7} f(x) = 0$.
   (b) Why did we require the sets A_n to be disjoint?

(4) (a) Using simpler trigonometric identities prove that, if $t = \tan\dfrac{x}{2}$, then

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}.$$

   (b) Prove that $\sin(10x)\sin(x) = \dfrac{1}{2}(\cos(9x) - \cos(11x))$.

(5) (a) Prove that if $\lim\limits_{x\to 0} f(x) = \ell$ then $\lim\limits_{x\to\infty} f(1/x) = \ell$.
   (b) Is the converse true? Namely, does $\lim\limits_{x\to\infty} f(1/x) = \ell$ imply that $\lim\limits_{x\to 0} f(x) = \ell$? Why or why not?

Solve the following questions but do not hand in your solutions:

- Spivak Chapter 5 Question 1 parts (i)–(vi) and Question 2. (Find limits.)
- Spivak Chapter 5 Question 3: choose two parts. (Find limits and prove.)
- Spivak Chapter 5 Question 8. (Relations between existence of limits of f, g, $f+g$.)
- Spivak Chapter 5 Question 10 parts (a)–(c). (Equivalent formulations of limit.)
- Spivak Chapter 5 Question 12. ($f \leq g$ implies $\lim f \leq \lim g$.)
- Spivak Chapter 5 Question 15 (i), (v), (viii). (Limit in terms of $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.)
- Spivak Chapter 5 Question 26. ("Bad" definitions of limits.)
- Spivak Chapter 5 Question 38. (Define what it means to diverge to ∞.)
- Spivak Chapter 5 Question 40. (Perimeter of regular polygon.)