MAT157 – ANALYSIS I, 2018–19. ASSIGNMENT 6.

Please re-read the appendix ("Uniform continuity") to Chapter 8 of Spivak's book, read the handout on uniform continuity on the course website, and read Chapter 9 ("Derivatives") of Spivak's book. Clear solutions to the following problems are due in the tutorial on Thursday November 8th.

(1) Let $f\colon (a,b) \to \mathbb{R}$ be a continuous function on the open interval (a,b). Assume that $\lim_{x\to a+} f(x) = \infty$ and that $\lim_{x\to b-} f(x) = \infty$. Let $y \in (a,b)$ be a point where f attains its minimum. Prove that, for every number c such that $c \geq f(y)$, there is $x \in (a,b)$ such that $f(x) = c$.

(2) Let $f\colon I \to \mathbb{R}$ and $g\colon J \to R$ be functions that are defined on interval I and J, respectively. Assume that $f(I) \subseteq J$; consider the composition $g \circ f\colon I \to \mathbb{R}$. Prove that if f is uniformly continuous on I and g is uniformly continuous on J then the composition $g \circ f$ is uniformly continuous on I.

(3) Solve Question 1(a) from the Appendix to Spivak's Chapter 8 (whether powers of x are uniformly continuous on $[0,\infty)$). Explain your solution.

(4) Let f be a rational function; write it as $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$ with $a_n, b_m \neq 0$. Prove that if $a_n/b_m > 0$ then there exists M such that $f(x) > 0$ for all $x > M$, and that if $a_n/b_m < 0$ then there exists M such that $f(x) < 0$ for all $x > M$.

(Hint: if $x > 0$ then the sign of $f(x)$ is the same as that of $\frac{x^{-n}}{x^{-m}} f(x)$.)

Please solve the following questions but do not hand in your solutions.

- Spivak Chapter 9 Problems 1(a), 2(a), 3, 4 (derivatives of $\frac{1}{x}$, $\frac{1}{x^2}$, $\sqrt{x}$, and x^n).
- Spivak Chapter 9 Problem 7 (not the derivative of a composition).
- Spivak Chapter 9 Question 12 (two cars on a road with speed limits).

For a function f with domain D, if f is bounded from above then we denote $\sup f := \sup\{f(x) \mid x \in D\}$, and if f attains a maximum then we denote $\max f := \max\{f(x) \mid x \in D\}$.

- Let $f\colon [a,b) \to \mathbb{R}$ be a monotone increasing continuous function on a half-closed interval $[a,b)$, and let d be a real number. (**Monotone increasing** means that for every x and y in I, if $x < y$ then $f(x) < f(y)$.)
  - Prove that if f is bounded from above and $d = \sup f$ then $\lim_{x\to b-} f(x) = d$.
  - Prove that if $\lim_{x\to b-} f(x) = d$ then the image of f is the half-closed interval $[f(a), d)$.
  - Prove that if f is not bounded from above then $\lim_{x\to b-} f(x) = \infty$.
  - Prove that if $\lim_{x\to b-} f(x) = \infty$ then the image of f is the ray $[f(a), \infty)$.