MAT157 – Analysis I, 2018–19. Assignment 10.

Please read the first page of Spivak's appendix to Chapter 12 ("Parametric representation of curves") (page 244), the notes on "length of a curve" on the course website, and Spivak's Chapter 13 ("Integrals"). Clear solutions to the following problems are due in the tutorial on Thursday January 17th.

(1) Define $\gamma\colon [-1,1] \to \mathbb{R}^2$ by $\gamma(t) = (\sqrt{1-t^2}, t)$. Prove that $\text{length}(\gamma) \geq 3$ (with more details than were provided in class or in the notes).

(2) (Invariance under reparametrization): Let $h\colon [a,b] \to [c,d]$ be a continuous function that is strictly monotone increasing and takes a to c and b to d, let $\gamma\colon [c,d] \to \mathbb{R}^2$ be a curve, and consider the curve $\gamma \circ h\colon [a,b] \to \mathbb{R}^2$.

  (a) Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. Let $h(P) := \{h(t_0), h(t_1), \ldots, h(t_n)\}$. Show that $h(P)$ is a partition of $[c,d]$ and that $\ell(\gamma, h(P)) = \ell(\gamma \circ h, P)$.

  (b) Show that the set $\{\ell(\gamma \circ h, P) \mid P \text{ is a partition of } [a,b]\}$ is a subset of the set $\{\ell(\gamma, Q) \mid Q \text{ is a partition of } [c,d]\}$. Conclude that, if γ is rectifiable, then $\gamma \circ h$ is rectifiable, and $\text{length}(\gamma \circ h) \leq \text{length}(\gamma)$.

  (c) Show that, if $\gamma \circ h$ is rectifiable, then γ is rectifiable, and $\text{length}(\gamma) \leq \text{length}(\gamma \circ h)$. (Hint: recall that h is invertible and that its inverse h^{-1} is continuous. Write $\gamma = (\gamma \circ h) \circ h^{-1}$ and apply part (b).)

  (d) Conclude that γ is rectifiable if and only if $\gamma \circ h$ is rectifiable, and, if so, then $\text{length}(\gamma) = \text{length}(\gamma \circ h)$.

(3) Let $f\colon [0,2] \to \mathbb{R}$ be given by $f(x) = \begin{cases} 10 & 0 \leq x < 1 \\ 100 & x = 1 \\ -5 & 1 < x \leq 2. \end{cases}$

Prove that f is Darboux integrable, and compute $\int_0^2 f$.

(4) Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let $\lambda \in \mathbb{R}$ be a real number. Suppose that f is Darboux integrable. Prove that λf is Darboux integrable and that $\int_a^b \lambda f = \lambda \int_a^b f$. (Hint: treat separately the cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.)

Optional:

Read Spivak's appendix to Chapter 12 ("Parametric representation of curves"). Solve the following questions but do not hand in your solutions.

- Spivak, appendix to Chapter 12, Problem 1 (Page 248). (Tangent to graph.)
- Spivak, appendix to Chapter 12, Problem 2 (page 248). ("Hidden corner.")
- Spivak, appendix to Chapter 12, Problem 9 (page 252). ("Mean value theorem" for planar curve.)
- Spivak, appendix to Chapter 12, Problem 10 (page 252). (Limit of function to $\mathbb{R}^2$ in terms if its components.)