# MAT157 – Analysis I, 2018–19. Assignment 12.

Please read the following materials.

- Spivak's Chapter 14 ("The fundamental theorem of calculus")
- The notes on "Length of a continuously differentiable curve" and on "Definitions of sine and cosine" that are posted on the course website.
- Spivak Chapter 15 ("The trigonometric functions"): please read until the middle of Page 305; then you can skip the three pages where Spivak gives definitions that differ from ours; then read from the middle of Page 308 until the end of the chapter.

Clear solutions to the following problems are due in class on Tuesday February 5th.

(1) For each of the following functions $F$, find an expression for the derivative $F'(x)$ without evaluating the integrals.
  (a) $F(x) = \int_a^{x^2} \cos^2(t)dt$.
  (b) $F(x) = \int_{-1}^x \left(\int_{-1}^y \frac{1}{1+t^4} dt\right) dy$.

(2) Determine whether the function $f(x) = \begin{cases} \sin(1/x) & \text{if } 0 < x \leq 2 \\ 0 & \text{if } x = 0 \end{cases}$ is integrable on $[0, 1]$.
  Justify briefly.

(3) Find $(f^{-1})'(0)$ if $f(x) = \int_{\pi/2}^x (1 + \sin(\cos(t)))dt$.

(4) Define $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$
  (a) Prove that $f$ is continuous at 0.
  (b) Prove that $f$ is differentiable at 0 and find $f'(0)$.
  (c) Prove that $f'$ is differentiable at 0 and find $f''(0)$.
  (You may use L'Hôpital's rule.)

(5) (a) Find $\int_0^1 \frac{1}{1+t^2} dt$.
  (b) Find $\int_0^\infty \frac{1}{1+t^2}$.
  (See the hint to Question 19 in Spivak's Chapter 15.)

Optional; do not hand in:

- Use properties of integrals and the definition of "limit" to prove the following "left" version of the first fundamental theorem of calculus:
  Let $f\colon [a, b] \to \mathbb{R}$ be an integrable function. Define $F\colon [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f$. Let $c \in (a, b]$. Suppose that $\lim_{\substack{x \to c \\ x < c}} f(x) = f(c)$. Then $\lim_{\substack{t \to c \\ t < c}} \frac{F(t) - F(c)}{t - c} = f(c)$.
- Let $f\colon [0, 1] \to \mathbb{R}$ be a bounded function. Define $g\colon [4, 5] \to \mathbb{R}$ by $g(x) = f(x - 4)$. From the definition of the Darboux integral, show that if $f$ is integrable then $g$ is integrable and $\int_4^5 g = \int_0^1 f$. (Hint: use ideas similar to those in Question 2 of Assignment 10.)