MAT157 – Analysis I, 2018–19. Assignment 16.

Please read Spivak Chapters 22 ("Infinite sequences").

Clear solutions to the following problems are due in class on Tuesday March 26th.

(1) (a) Let $n \in \mathbb{N}$. Let f be $(n+1)$ times differentiable on $[a,b]$. Suppose that its $n+1$st derivative satisfies $|f^{(n+1)}| \leq M$ everywhere on $[a,b]$. Let $P_{n,a}(x)$ be the Taylor polynomial of degree n of f at a. Use the Lagrange form of the remainder to show that $|f(x) - P_{n,a}(x)| \leq \frac{M(x-a)^{n+1}}{(n+1)!}$.

(b) Use the above estimate to calculate $\sin \frac{1}{10}$ up to an accuracy of $\frac{1}{100{,}000}$. Explain your reasoning.

(c) The answer that you got in (b) should not surprise you. Why?

(2) Let f and g be infinitely differentiable functions. Let $a_k = \frac{f^{(k)}(a)}{k!}$ and $b_m = \frac{g^{(m)}(a)}{m!}$ be the coefficients of their Taylor polynomials at a. Find the coefficients of the Taylor polynomials of the product fg at a. Justify your answer.

(Hint: In the proof, it will be easier to use a characterization of Taylor polynomials than the definition of Taylor polynomials.)

(3) Prove that if $x \leq 0$ then the remainder term $R_{n,0}$ for e^x satisfies $|R_{n,0}| \leq \frac{|x|^{n+1}}{(n+1)!}$.

(4) Prove that if $f''(a)$ exists then $f''(a) = \lim_{h \to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}$. (Hint: use the Taylor polynomial $P_{2,a}(x)$ of degree 2 for f at a and substitute $x = a+h$ and $x = a-h$.)

(5) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is three (or more) times differentiable at the point 5 and that $f(x) \leq f(5)$ for all $x \in \mathbb{R}$. Suppose that $f''(5) = 0$. Prove that $f'''(5) = 0$.

Solve the following questions. Do not hand in your solutions.

- Spivak Chapter 20, Page 431, Problem 1: (ii), (viii), (x). (Find Taylor polynomials.)
- Spivak Chapter 20, Page 431, Problem 2: (iv). (Manipulate polynomials.)
- Spivak Chapter 20, Page 433, Question 8. (Taylor polynomial of $\sin(x^2)$ and consequences.)
- Prove that Euler's constant e $(= \exp(1))$ is equal to 2.72 up to an error of at most $\frac{1}{100}$.