MAT157 – Analysis I, 2018–19. Assignment $16\frac{1}{2}$.

Please read Spivak Chapters 23 ("Infinite series").

Please read Spivak Chapter 25 ("Complex numbers"); you can skip the part about solving a cubic equation. Please also watch the 14-minute movie about complex numbers in Chapter 5 of Étienne Ghys's "Dimensions" series, here: `https://www.youtube.com/watch?v=T-c8hvMXENo&list=PL3C690048E1531DC7&index=5`.

Please solve the following questions in preparation for the April 2nd term test. You do not need to submit your solutions.

("Heads up": the final exam will also cover Chapter 24.)

- Spivak Chapter 22, Problem 1, parts (iv), (vi), (viii).
  (Limits.)
- Spivak Chapter 23 Problem 1 (page 489): do three parts. (Infinite series.)
- Spivak Chapter 23, Problem 7 (page 492), all parts.
  (Decimal expansions.)
  In Part (b), please assume that $0 \leq x < 1$.
  Hint: $s_N \leq x \leq s_N + \frac{1}{10^N}$.
- Prove: if $a_n \to \ell$ and $a_n \to \ell'$, then $\ell = \ell'$.
  Prove: if $a_n \to \ell$ and $b_n \to \ell'$ and $a_n \leq b_n$ for all n, then $\ell \leq \ell'$.
  Prove: if $a_n \to \ell$ and $b_n \to \ell'$, then $a_n + b_n \to \ell + \ell'$.
  Prove: if $a_n \to \ell$ and $b_n \to \ell'$, then $a_n \cdot b_n \to \ell \cdot \ell'$.
  Conclude: if $a_n \to \ell$ and $b_n \to \ell'$ and $\ell' \neq 0$, then $a_n/b_n \to \ell/\ell'$. (Hint: $x \mapsto 1/x$ is continuous.)
- Prove: if $\{a_n\}$ is non-increasing and bounded from below, then $\{a_n\}$ converges.
  (The book proves the non-decreasing case.)
- Write the negation of $\lim_{n\to\infty} a_n = \ell$ without using the word "limit".
- The meaning of the sentence $\lim_{n\to\infty} a_n = \ell$ begins with "For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that". Obtain a new sentence by changing the beginning to "There exists $N \in \mathbb{N}$ such that for all $\epsilon > 0$". Show that the new sentence has a different meaning by giving an example of a sequence that satisfies one of these sentences but not the other.
- Spivak Chapter 22 Problem 9: do three parts. (Mostly uses Riemann sums.)
- Prove that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.
  (Hint: Show that $\lim_{t\to 0+} \frac{\log(1+t)}{t} = 1$. Write $\left(1 + \frac{x}{n}\right)^n = e^{x\frac{\log(1+t)}{t}}$ with $t = \frac{x}{n}$.)
  (Anecdote: google "math stack exchange 1/e cookies".)