Please follow the instructions.

Please answer all the questions in the booklet that is provided. You can answer the questions in any order, but please use separate pages for different questions, and mark the question number at the top corner of the page.

The nine questions have equal weight. Please use clear handwriting and clear language; unclear or irrelevant text may cause you to lose points. A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section.

Good luck!

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function.

(a) Define "$\lim_{x\to\infty} f(x) = \ell$" (where ℓ is some real number).

(b) Use the definition to prove that if f is non-decreasing and bounded from above then there is an ℓ such that $\lim_{x\to\infty} f(x) = \ell$.

(2) (a) Find the minimum and maximum values of the function $x \mapsto \dfrac{1-x}{1+x^2}$ on the interval $[0,3]$.

(b) Sketch, to the best of your understanding, the graph of this function on the interval $[0,3]$.

Show your work.

(3) Let $f(x) = \displaystyle\int_{\pi/2}^{x} \left(2 + \cos\left(\frac{\pi}{4}\sin t\right)\right) dt$. Let f^{-1} be its inverse function and $(f^{-1})'$ its derivative. Find $(f^{-1})(0)$ and $(f^{-1})'(0)$; simplify your answer if possible.

(4) Use the substitution $t = \tan\frac{x}{2}$ to transform the integral $\int (\sin x)^5 (\cos x)^7\, dx$ into an integral of a rational function in the variable t. Do not simplify nor evaluate the resulting integral.

(5) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n \in \{0,\ldots,9\}$ for all n.

(a) Show that $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges.

(b) Let $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$. Let $s_N = \sum_{n=1}^{N} \frac{a_n}{10^n}$.
Show that $s_N \le x \le s_N + \frac{1}{10^N}$.

continued on page 4

(6) Evaluate the following limit:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{e}+\sqrt[n]{e^2}+\ldots+\sqrt[n]{e^{2n}}}{n}\right).$$

Do this by interpreting the sum as a Riemann sum for a suitable integral; specify the function, the interval, and the partition.

(7) (a) For which values of x does the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$ converge?

(b) Find a function $f(x)$ to which this series converges.

(8) For each of the following statements, determine if it is true or false. Do not explain.

(a) "For any continuous function $f\colon \mathbb{R}\to\mathbb{R}$, if for each $x\in\mathbb{R}$ there exists $n\in\mathbb{N}$ such that $nf(x)\in\mathbb{N}$, then $f(0)=f(1)$."

(b) "For any function $f\colon [0,\infty)\to\mathbb{R}$, if f is positive and continuous and $\lim_{x\to\infty} f(x)=0$, then there exists $c\geq 0$ such that $f(c)\geq f(x)$ for all $x\geq 0$."

(c) "For any function $f\colon [0,1]\to\mathbb{R}$, if f is Darboux integrable, then the function $x\mapsto \displaystyle\int_0^x f(t)dt$ is uniformly continuous on $[0,1]$."

(d) "If $a_n\geq 0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty} na_n$ exists then the series $\sum a_n$ converges."

(9) (a) Give an example of a function $f\colon [0,1]\to\mathbb{R}$ that is bounded but not Darboux integrable.

(b) Give an example of a function $f\colon [0,1]\to\mathbb{R}$ that is Darboux integrable but is not the derivative of any differentiable function on $[0,1]$.

(c) Give an example of a function $f\colon \mathbb{R}\to\mathbb{R}$ that is bounded and continuous but not uniformly continuous.

(d) Give an example of a sequence of continuous functions $f_n\colon [0,1]\to\mathbb{R}$ such that $x\mapsto \lim_{n\to\infty} f_n(x)$, for $x\in[0,1]$, is well defined but is not continuous.

Do not explain.