Please answer all the questions in the booklet that is provided. You can answer the questions in any order, but please use separate pages for different questions, and mark the question number at the top corner of the page.

Please use clear handwriting and clear language; unclear or irrelevant text may cause you to lose points.

Show your work.

The nine questions have equal weight. A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section.

Good luck!

(1) Let A and B be non-empty sets of real numbers. Assume that every element of A is smaller than every element of B. Prove that there exists a real number ℓ such that $a \le \ell \le b$ for all $a \in A$ and $b \in B$.

(2) Find the maximum and minimum values of the function $\dfrac{x+1}{x^2+1}$ on the interval $[-1, \frac{1}{2}]$.

(3) Sketch, to the best of your understanding, the graph of the function $f(x) = \dfrac{e^{-x}}{x}$.

(4) Let $F(x) = \int_5^{e^x} (\log t)^2 dt$.

(a) At which points x is the function $F(x)$ well defined?

(b) Find $F'(x)$. Simplify if you can.

(c) The function F is invertible. Why?

Let F^{-1} be the inverse function.

(d) Find $F^{-1}(0)$.

(e) Find $(F^{-1})'(0)$. Simplify if you can.

(5) Calculate $\int_4^{\infty} e^{-\sqrt{x}} dx$.

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(6) Compute the volume of the body of revolution that is obtained by rotating about the horizontal axis the triangle with vertices $(0,4)$, $(1,5)$, and $(2,4)$.

(7) (a) Let $f(x)$ be a function that is n times differentiable at the point 0. Let $P(x)$ be the order n Taylor polynomial of the function $f(x)$ at the point 0. Without computing derivatives, prove that the order $2n$ Taylor polynomial of the function $f(x^2)$ at 0 is $P(x^2)$.

(b) Recall that the Taylor series of the function e^x is $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ (don't prove this fact). For each $k \in \mathbb{N}$, find an expression for the kth derivative of the function $x \mapsto e^{x^2}$ at the point 0.

(8) (a) For which values of x does the power series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ converge?

(b) Find a function $f(x)$ to which this series converges.

(9) For each of the following statements, determine if it is true or false, and add an explanatory sentence or diagram or formula.

(a) "For every differentiable function $f\colon (0,\infty) \to \mathbb{R}$, if $f'(x) > 0$ for all $x \in (0,\infty)$, then $\lim\limits_{x\to\infty} f(x) = \infty$."

(b) "The improper integral $\int\limits_0^1 \frac{1}{\sqrt{1-x^2}} dx$ converges."

(c) "If a function $f(x)$ is differentiable to all orders at the point 0 and $P_n(x)$ is the Taylor polynomial of order n of f at the point 0 then there exists $\epsilon > 0$ such that for all $x \in (-\epsilon, \epsilon)$ we have $\lim\limits_{n\to\infty} P_n(x) = f(x)$."

(d) "Let $f\colon [-1,1] \to \mathbb{R}$ be an integrable function, and define $F(x) = \int\limits_{-1}^{x} f(t)dt$ for all $x \in [-1,1]$. If $F(x)$ is differentiable at the point 0, then $f(t)$ is continuous at the point 0."

(e) "For every positive number A there exists a positive number x such that $\int_0^x \frac{1+(\sin t)^4}{1+t} dt = A$."