MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 3, FOR CREDIT

(1) We say that a subset A of $\mathbb{R}$ is **dense** in $\mathbb{R}$ if every open interval contains an element of A.

(a) Consider the statement

The set of rational numbers is dense in $\mathbb{R}$.

Rewrite this statement without using the word "dense" and without using the word "interval".

(b) Prove that every interval of length > 1 contains an integer.
(Hint: given an interval (x, y), take the largest integer l that is smaller than y. Why does it exist? Why does it do the job?)

(c) Prove that the set of rational numbers is dense in $\mathbb{R}$.
(Hint: given an interval (x, y), for large enough n the interval (nx, ny) has length > 1. Why?)

(d) Prove that the set of irrational numbers is dense in $\mathbb{R}$.
(Hint: given an interval (x, y), consider the interval $(\sqrt{2}x, \sqrt{2}y)$ and apply (c).)

(2) Let A and B be subsets of $\mathbb{Q}$. Assume:

(i) A and B are non-empty, and their union is $\mathbb{Q}$ (i.e., every rational number is in A or in B).

(ii) If $a \in A$ and $b \in B$, then $a < b$.

Prove that there exists a real number γ such that, for every rational number x, if $x < \gamma$ then $x \in A$ and if $x > \gamma$ then $x \in B$.
(Hint: take $\gamma := \sup A$. Why does it exist?.)

(3) Fix a set A of real numbers. Define $-A := \{-x \mid x \in A\}$.

(a) Give the definitions of a **lower bound** for A and of the **greatest lower bound** for A (which is denoted $\inf A$).

(b) Prove that if β is a lower bound for A then $-\beta$ is an upper bound for $-A$.

(c) Prove that if $\beta = \inf A$ then $-\beta = \sup(-A)$.

(d) Prove that if A is non-empty and has a lower bound then A has a greatest lower bound.
(Use the least upper bound axiom.)