MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 6

(1) (i) State a theorem about the image of a bounded increasing continuous function on the ray $(-\infty, a]$.

(ii) State a theorem about the image of an unbounded increasing continuous function on the ray $(-\infty, a]$.

(iii) State a theorem about the image of a bounded decreasing continuous function on the ray $(-\infty, a]$.

(iv) State a theorem about the image of an unbounded decreasing continuous function on the ray $(-\infty, a]$.

Do not submit proofs for any of these theorems.

(2) Let f and g be continuous functions on $[a, b]$. Suppose that $f(a) < g(a)$ but $f(b) > g(b)$. Prove that there exists an x between a and b such that $f(x) = g(x)$.

(Hint: it's short.)

(3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $f(x)$ is irrational for all x. Prove that f is constant.

(Hint: assume that f is not constant and use the intermediate value theorem.)

(4) Let f be a rational function; write it as $f(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$ with $a_n, b_m \neq 0$.

Prove that if $a_n/b_m > 0$ then there exists M such that $f(x) > 0$ for all $x > M$, and that if $a_n/b_m < 0$ then there exists M such that $f(x) < 0$ for all $x > M$.

(Hint: if $x > 0$ then the sign of $f(x)$ is the same as that of $\frac{x^{-n}}{x^{-m}} f(x)$.)