MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 7

(1) Let $f\colon I \to \mathbb{R}$ and $g\colon J \to \mathbb{R}$ be functions that are defined on interval I and J, respectively. Assume that $f(I) \subseteq J$; consider the composition $g \circ f\colon I \to \mathbb{R}$. Prove that if f is uniformly continuous on I and g is uniformly continuous on J then the composition $g \circ f$ is uniformly continuous on I.

(2) (Adapted from Spivak's Chapter 9 Problem 12:)

Imagine a road on which the speed limit is specified at every single point. In other words, there is a certain function L such that the speed limit x kilometers from the beginning of the road is $L(x)$ kilometers/hour. Two cars, A and B, are driving along this road; car A's position at time t is $a(t)$, and car B's position at time t is $b(t)$.

(a) What equation expresses the fact that car A always travels at the speed limit? (Hint: the answer is *not* $a'(t) = L(t)$.)

(b) Suppose that A always goes at the speed limit, and that B's position at time t is A's position at time $t-1$. Show that B is also going at the speed limit at all times.

(c) Suppose B always stays a constant distance behind A. Under what conditions will B still travel at the speed limit?

(3) Let $f(x) = 1/x$.

(a) Find the equation for the tangent line to the graph of f at the point $(7, 1/7)$.

(b) Find the linear approximation for $f(x)$ near the point $x = 5$. Write it as "$L(x) = \ldots$".

(4) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function (i.e., a function that is differentable at each point). Define a new function by

$$g(x) = \begin{cases} (f(x))^2 & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0\,. \end{cases}$$

Let c be a point in $\mathbb{R}$.

(a) Prove that if $f(c) > 0$ then g is differentiable at c.

(b) Prove that if $f(c) = 0$ then g is differentiable at c.