MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 14

(1) Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map $R(x, y) = (-y, x)$. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a continuous curve. Consider the curve $R \circ \gamma\colon [a, b] \to \mathbb{R}^2$, given by $(R \circ \gamma)(t) = R(\gamma(t))$.

Recall that, for any partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$, we have $\ell(P, \gamma) = \sum_j \|\gamma(t_{j-1}) - \gamma(t_j)\|$.

(a) Prove that, for every partition P of $[a, b]$, we have $\ell(P, R \circ \gamma) = \ell(P, \gamma)$.

(b) Prove that the sets

$$\{\ell(P, \gamma) \mid P \text{ is a partition of } [a, b]\}$$

and

$$\{\ell(P, R \circ \gamma) \mid P \text{ is a partition of } [a, b]\}$$

are equal.

(c) Prove that $R \circ \gamma$ is rectifiable if and only if γ is rectifiable and, if so, then $\text{length}(R \circ \gamma) = \text{length}(\gamma)$.

(d) Write $\gamma(t) = (x(t), y(t))$, and assume that the functions x and y are of type C^1. Use the formula for length as the integral of speed to give another proof that $\text{length}(R \circ \gamma) = \text{length}(\gamma)$.

(2) Determine which of the following improper integrals converges. Justify your answer.

(a) $\displaystyle\int_0^1 \frac{dx}{x\sqrt{2-x}}$

(b) $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x-2}}$

(3) Sketch rough graphs of the following two functions. Do not attempt to draw the graphs to scale. Have each graph show the domain of definition, the minimum and maximum points and values of the function, the regions where the function increases or decreases, and the behaviour of the function near points where it is not defined (if any) and as $x \to \pm\infty$; show the x- and y- intercepts (if any). Show your work.

(a) $f(x) = x^{100} \exp(-x)$

(b) $f(x) = x^{100} \exp(-x^2)$