MA157 - ANALYSIS 1, 2020-2021. ASSIGNMENT 15

- (1) For each of the following statements, determine if it is true or false. Explain with a brief sentence or diagram or formula. You do not need to give a complete proof.
 - (a) Let

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Then the improper integral $\int_0^\infty f(x) dx$ converges.

- (b) The curve $\gamma(t) := (1, \sqrt[3]{t})$, for $t \in [-1, 1]$, is rectifiable.
- (c) The following function is differentiable at x = 0:

$$f(x) := \begin{cases} \frac{1}{x} e^{-1/x} & \text{if } x > 0\\ 0 & \text{if } x \le 0 \end{cases}$$

- (d) For a continuous function $f: (0,1] \to \mathbb{R}$, if f is bounded on every closed subinterval of (0,1], then f is bounded.
- (2) Define $f: (0, \frac{\pi}{2}) \to \mathbb{R}$ by

$$f(x) := \int_{\frac{\pi}{3}}^{x} \frac{1}{1 - \sin^4 t}.$$

- (a) Why is f one-to-one?
- Let $g := f^{-1}$ be the inverse function.
- (b) Find g'(0).
- (c) What can you say about the image of g?
- (d) What can you say about the domain of g?
- (3) (a) State a criterion, in terms of upper and lower Darboux sums, that is necessary and sufficient for a function to be Darboux integrable.
 - (b) Let $f: [-1,1] \to \mathbb{R}$ be a bounded function that is Darboux integrable. Let $g(x) := \max\{10, f(x)\}$. Use the criterion from (a) to prove that g is Darboux integrable.
- (4) Let $f: (-2,2) \to \mathbb{R}$ and $g: (-2,2) \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 0 & \text{otherwise .} \end{cases} \quad \text{and} \quad g(x) := \int_0^x f(t) dt.$$

For which values of x is g'(x) = f(x)?