MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 16

(1) Find a number c such that the function $f(x) := \begin{cases} \dfrac{1-\cos x}{x^2} & x \neq 0 \\ c & x = 0 \end{cases}$ is continuous. Justify.

(2) Find $\displaystyle\int_1^\infty \frac{dt}{1+t^2}$. Show your work. (See Question 19 in Spivak's Chapter 15.)

(3) Using the values of the sine and cosine functions at 0 and $\pi/2$ and the facts that $\frac{d}{d\alpha}\sin\alpha = \cos\alpha$ and $\frac{d}{d\alpha}\cos\alpha = -\sin\alpha$ for all α, prove that, for all α, we have $\cos(\alpha+\pi/2) = -\sin\alpha$. (Hint: if $f + f'' \equiv 0$ and $f(0) = f'(0) = 0$, then $f \equiv 0$.)

(4) Any function that is obtained from constants, $\sin(x)$, and $\cos(x)$ by additions, subtractions, multiplications, and quotients, has an elementary primitive. Indeed, the evil substitution $t = \tan\frac{x}{2}$ transforms the integral of this function into the integral of a rational functions. Find $\int \frac{dx}{2+\cos x}$ using this method. (Hint: see Chapter 19 Problem 12)

(5) Let $I_n := \int x^n e^x dx$. Find a reduction formula that expresses I_n in terms of I_{n-1} for every positive integer n. (cf. Spivak Chapter 19 Questions 21,22).

(6) (From past final exams at St.George:)
   (a) Evaluate $\int \frac{x^2+2}{x^4+2x^2+1}dx$.
   (b) Evaluate $\int_0^1 \sqrt{x}\arctan(\sqrt{x})dx$.

Here are a few extra questions. Do not hand in their solutions.

- (From a past final exam at St.George:) Let $f(x) := x^2 - x\sin x - \cos x$. Prove that the equation $f(x) = 0$ has exactly two real solutions.
- (From a past final exam at St.George:) Prove that, for all $x, y \in \mathbb{R}$, we have $|\arctan(x) - \arctan(y)| \leq |x-y|$, with equality if and only if $x = y$.
- Using the definition of the cosine function in the handout, and using the facts (which are proved in the handout) that $\frac{d}{d\alpha}\sin(\alpha) = \cos(\alpha)$ for all α and that the sine function is continuous, prove:
  - If $\alpha \in (-\pi/2 + k\pi, \pi/2 + k\pi)$ for an even integer k, then $\frac{d}{d\alpha}\cos\alpha = -\sin\alpha$.
  - If $\alpha \in (-\pi/2 + k\pi, \pi/2 + k\pi)$ for an odd integer k, then $\frac{d}{d\alpha}\cos\alpha = -\sin\alpha$.
  - If $\alpha = \pi/2 + k\pi$ for any integer k, then $\frac{d}{d\alpha}\cos\alpha = -\sin\alpha$.