MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 18

(1) Prove that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x)$.

(Hint: show that $\lim\limits_{\substack{t\to 0\\ t>0}} \frac{\log(1+t)}{t} = 1$. Write $\left(1+\frac{x}{n}\right)^n = \exp\left(x\frac{\log(1+t)}{t}\right)$ with $t=\frac{x}{n}$.)

(Anecdote: google "math stack exchange $1/e$ cookies".)

(2) (From a past final exam:)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n \in \{0,\ldots,9\}$ for all n.

(a) Show that $\sum\limits_{n=1}^{\infty}\frac{a_n}{10^n}$ converges.

(b) Let $x=\sum\limits_{n=1}^{\infty}\frac{a_n}{10^n}$. Let $s_N=\sum\limits_{n=1}^{N}\frac{a_n}{10^n}$.

Show that $s_N \le x \le s_N + \frac{1}{10^N}$.

(3) Find

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right)$$

by expressing the terms of this sequence as Riemann sums.
(See Spivak Chapter 22 Problem 9.)

(4) (From a past final exam :)

For what values of α does the series $\sum\limits_{n=2}^{\infty}\frac{1}{n(\log n)^{\alpha}}$ converge?
For what values of α does it diverge?
Explain.

(5) For each of the following statements, determine if it is true or false, and add a **brief** one-sentence explanation (not a complete proof!).

(a) "The curve $\gamma(t)=(t^{1/3},t^{2/3})$, for $t\in[0,1]$, is rectifiable".

(b) "Let $f\colon[0,1]\to\mathbb{R}$ be a bounded function. If for each $0<\alpha<1$ the function f is Darboux integrable on the interval $[\alpha,1]$, then the function f is Darboux integrable on the interval $[0,1]$."