MA157 – ANALYSIS 1, 2020–2021. ASSIGNMENT 19

(1) (From the UTSG 2015 final exam:)

(a) In what interval does the power series $f(x) = \sum_{n=1}^{\infty} nx^n$ converge?

(b) Find an expression for $f(x)$ as a rational function on its interval of convergence. Justify.

(2) Find the Taylor polynomial of order 2 of the function $f(x) = x^3$ at the point $\gamma = 1$.

(3) (a) Let $n \in \mathbb{N}$. Let f be $(n+1)$ times differentiable on $[0,1]$. Suppose that its $n+1$st derivative satisfies $|f^{(n+1)}| \leq M$ everywhere on $[0,1]$. Let $P_n(x)$ be the Taylor polynomial of degree n of f at the point 0. Use the Lagrange form of the remainder to show that $|f(x) - P_n(x)| \leq \frac{Mx^{n+1}}{(n+1)!}$ for all $x \in [0,1]$.

(b) Using the above estimate, and the fact that Euler's constant e $(= \exp(1))$ satisfies $0 < e < 4$, show that, if $n = 6$, the error term $R_n(x)$ for the nth order Taylor polynomial for $\exp(x)$ at 0 satisfies $|R_n(x)| \leq 1/1,000$ for all x in the interval $[0,1]$.

(c) Using the above items, prove that Euler's constant e is equal to 2.72 up to an error of at most 1/100.

(4) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is four times differentiable at the point 1, satisfies $f'(1) = f''(1) = 0$ and $f^{(4)}(1) \neq 0$, and is strictly monotone increasing near the point 1. Prove that $f^{(3)}(1) \neq 0$.