**APPROXIMATIONS.**

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**Addition:** Fix real numbers $a$ and $b$. Fix $\epsilon > 0$. We claim that there exists $\delta > 0$ such that, for all $x$ and $y$, if $|x - a| < \delta$ and $|y - b| < \delta$ then $|(x + y) - (a + b)| < \epsilon$.

(Thus, we can guarantee that $x + y$ be arbitrarily close to $a + b$ by requiring that $x$ and $y$ be sufficiently close to, respectively, $a$ and $b$.)

*Discussion:* Express $(x + y) - (a + b)$ in terms of $x - a$ and $y - b$ and apply the triangle inequality:

$$|(x + y) - (a + b)| = |(x - a) + (y - b)| \leq |x - a| + |y - b|. \tag{1}$$

*Solution:* Take $\delta = \frac{1}{2}\epsilon$.

*Proof that it works:* Suppose that $|x - a| < \delta$ and $|y - b| < \delta$. Then the right hand side of (1) is $< 2\delta$, which is equal to $\epsilon$. So the left hand side is $< \epsilon$, as required.

**Multiplication:** Fix real numbers $a$ and $b$. Let $\epsilon > 0$. We claim that there exists $\delta > 0$ such that, for all $x$ and $y$, if $|x - a| < \delta$ and $|y - b| < \delta$, then $|xy - ab| < \epsilon$.

(Thus, we can guarantee that $xy$ be arbitrarily close to $ab$ by requiring that $x$ and $y$ be sufficiently close to, respectively, $a$ and $b$.)

*Discussion:* Express $xy - ab$ in terms of $x - a$ and $y - b$ and apply the triangle inequality:

$$|xy - ab| = |(x - a)(y - b) + a(y - b) + (x - a)b| \leq |x - a||y - b| + |a||y - b| + |b||x - a| \tag{2}$$

Suppose that $|x - a| < \delta$ and $|y - b| < \delta$. Then the second and third summands on the right hand side of (3) are, respectively, $\leq |a|\delta$ and $\leq |b|\delta$. If also $\delta \leq 1$, then the first summand on the right hand side of (3) is $< \delta$, and the entire right hand side of (3) is $< (1 + |a| + |b|)\delta$. Note that $1 + |a| + |b|$ is positive.

*Solution:* Take $\delta = \min\{1\,,\ \frac{1}{1+|a|+|b|}\epsilon\}$.

*Proof that it works:* Suppose that $|x - a| < \delta$ and $|y - b| < \delta$. Then the right hand side of (1) is $< (1 + |a| + |b|)\delta$, which is $\leq \epsilon$. So the left hand side is $< \epsilon$, as required.

**Inverse:** Fix a non-zero real number $a$. Let $\epsilon > 0$. We claim that there exists $\delta > 0$ such that, for all $x$, if $|x - a| < \delta$, then $|\frac{1}{x} - \frac{1}{a}| < \epsilon$.

(Thus, we can guarantee that $\frac{1}{x}$ be arbitrarily close to $\frac{1}{a}$ by requiring that $x$ be sufficiently close to $a$.)

*Discussion:* Express $\frac{1}{x} - \frac{1}{a}$ in terms of $x - a$:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a-x}{xa} \right| = \frac{|a-x|}{|x||a|}. \tag{3}$$

Suppose that $|x - a| < \delta$. By the reverse triangle inequality, $|x| > |a| - \delta$. So if $\delta \leq \frac{1}{2}|a|$, then $|x| > \frac{1}{2}|a|$; the denominator of the right hand side of (3) is then $> \frac{1}{2}|a|^2$, and the entire right hand side of (3) is then $< \frac{\delta}{\frac{1}{2}|a|^2}$.

*Solution:* Take $\delta = \min\{\frac{1}{2}|a|\,,\ \frac{1}{2}|a|^2\epsilon\}$.

*Proof that it works:* Suppose that $|x - a| < \delta$. Then the right and side of (3) is $< \frac{\delta}{\frac{1}{2}|a|^2}$, which is $\leq \epsilon$. So the left hand side is $< \epsilon$, as required.

**Square root:** Fix a positive real number $a$. Let $\epsilon > 0$. We claim that there exists $\delta > 0$ such that, for all $x$, if $|x - a| < \delta$, then $|\sqrt{x} - \sqrt{a}| < \epsilon$.

(Thus, we can guarantee that $\sqrt{x}$ be arbitrarily close to $\sqrt{a}$ by requiring that $x$ be sufficiently close to $a$.)

*Discussion:* To guarantee that $\sqrt{x}$ is well defined whenever $|x - a| < \delta$, we require $\delta \leq a$.

To express $\sqrt{x} - \sqrt{a}$ in terms of $x - a$, multiply and divide it by $\sqrt{x} + \sqrt{a}$:

$$\left|\sqrt{x} - \sqrt{a}\right| = \left| \frac{x-a}{\sqrt{x}+\sqrt{a}} \right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}. \tag{4}$$

Suppose that $|x - a| < \delta$. The denominator of the right hand side of (4) is $\geq \sqrt{a}$, so the entire right hand side of (4) is $< \frac{\delta}{\sqrt{a}}$.

*Solution:* Take $\delta = \min\{\, a\,,\ \sqrt{a} \cdot \epsilon\,\}$.

*Proof that it works:* Suppose that $|x - a| < \delta$. Then the right hand side of (4) is $< \frac{\delta}{\sqrt{a}}$, which is $\leq \epsilon$. So the left hand side is $< \epsilon$, as required.