Postulates for the real numbers, and some consequences.

We have the following structure.

- A set, denoted $\mathbb{R}$, whose elements we call the **real numbers**.
- Two distinguished elements of $\mathbb{R}$, denoted $\mathbf{0}$ and $\mathbf{1}$.
- An operation, called **addition**, which associates to any pair of real numbers (a, b) a third real number, denoted $\boldsymbol{a+b}$.
- An operation, called **multiplication**, which associates to any pair of real numbers (a, b) a third real number, denoted $\boldsymbol{a \cdot b}$.
- A distinguished subset P of $\mathbb{R}$, whose elements we call the **positive numbers**.

Postulates (P1)–(P4) are about addition.

(P1) (addition is associative):
for all $a, b, c \in \mathbb{R}$, $\quad a + (b + c) = (a + b) + c.$

(P2) (0 is neutral for addition):
for all $a \in \mathbb{R}$, $\quad a + 0 = 0 + a = a.$

Theorem (Uniqueness of the neutral element for addition)**.**
If for all $a \in \mathbb{R}$ we have $a + x = x + a = a$, then $x = 0$.

(P3) (Existence of an additive inverse):
For every $a \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $a + x = x + a = 0$.

Theorem (Uniqueness of the additive inverse)**.**
If $a + x = x + a = 0$ and $a + y = y + a = 0$, then $x = y$.

Notation. We denote the x such that $a + x = x + a = 0$ by $\boldsymbol{-a}$.

Theorem. $\quad -(-a) = a. \quad -(a + b) = (-b) + (-a). \quad -0 = 0.$

Notation. $a - b := a + (-b)$.

Theorem. $\quad a - a = 0. \quad (a + b) - c = a + (b - c). \quad -(a - b) = b - a.$

(P4) (addition is commutative):
for all $a, b \in \mathbb{R}$, $\quad a + b = b + a.$

Postulates (P5)–(P8) are about multiplication.

(P5) (multiplication is associative):
for all $a, b, c \in \mathbb{R}$, $\quad a \cdot (b \cdot c) = (a \cdot b) \cdot c.$

(P6) (1 is neutral for multiplication and is different from 0):
$1 \neq 0$, and, for all $a \in \mathbb{R}$, $\quad a \cdot 1 = 1 \cdot a = a.$

Theorem (Uniqueness of the neutral element for multiplication)**.**
If for all $a \in \mathbb{R}$ we have $a \cdot x = x \cdot a = a$, then $x = 1$.

(P7) (Existence of a multiplicative inverse):
for every $a \neq 0$ there exists an x such that $a \cdot x = x \cdot a = 1.$

Theorem (Uniqueness of the multiplicative inverse)**.**
If $a \neq 0$ *and* $a \cdot x = x \cdot a = 1$ *and* $a \cdot y = y \cdot a = 1$*, then* $x = y$*.*

Notation. For $a \neq 0$, we denote the x such that $a \cdot x = x \cdot a = 1$ by $\boldsymbol{a^{-1}}$.

Theorem. $(a^{-1})^{-1} = a$. $(ab)^{-1} = (b^{-1})(a^{-1})$. $1^{-1} = 1$.

Notation. For $b \neq 0$, $a/b := a \cdot b^{-1}$.

Theorem. $a/a = 1$. $(a \cdot b)/c = a \cdot (b/c)$. $(a/b)^{-1} = b/a$.

(P8) (Multiplication is commutative):
for all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$.

Postulate (P9) relates addition to multiplication.

(P9) (Distributive law):
for all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Theorem.

- *For all* a, $a \cdot 0 = 0$.
- *For all* a, b, $(-a) \cdot b = -(a \cdot b)$.
- *For all* a, b, $(-a) \cdot (-b) = a \cdot b$.

Postulates (P10)–(P12) are about positive numbers. They imply properties of the ordering of real numbers.

(P10) (Trichotomy): for every $x \in \mathbb{R}$, exactly one of the following three cases holds:
$x = 0$, $x \in P$, $-x \in P$.

(P11) (Positive numbers are closed under addition):
for every $x, y \in \mathbb{R}$, if $x, y \in P$, then $x + y \in P$.

(P12) (Positive numbers are closed under multiplication):
for every $x, y \in \mathbb{R}$, if $x, y \in P$, then $x \cdot y \in P$.

Definition.

We say that $\boldsymbol{a > b}$ if $a - b \in P$.
We say that $\boldsymbol{a < b}$ if $b > a$.
We say that $\boldsymbol{a \geq b}$ if $a > b$ or $a = b$.
We say that $\boldsymbol{a \leq b}$ if $a < b$ or $a = b$.

Theorem (Characterization of positive numbers)**.** $a \in P$ *if and only if* $a > 0$.

Theorem (Trichotomy)**.** *For every* $a, b \in \mathbb{R}$*, exactly one of the following three cases holds:* $a > b$, $a = b$, $a < b$.

Theorem (Antisymmetry of $\leq$)**.**
For every $a, b \in \mathbb{R}$*, if* $a \leq b$ *and* $b \leq a$*, then* $a = b$*.*

Theorem (Transitivity of $\leq$)**.**
For every $a, b, c \in \mathbb{R}$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Also: if $a < b$ and $b \leq c$ then $a < c$; if $a \leq b$ and $b < c$ then $a < c$.

Theorem. *If $x \neq 0$, then $x \cdot x > 0$.*

Corollary. $1 > 0$.

Corollary. $-1 < 0$.

Theorem. *There does not exist an x such that $x \cdot x = -1$.*

Theorem. *If $a < b$, then $a + c < b + c$.*

Theorem. *If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.*

Theorem.

- *If $a > 0$ and $b > 0$, then $a \cdot b > 0$.*
- *If $a < 0$ and $b < 0$, then $a \cdot b > 0$.*
- *If $a > 0$ and $b < 0$, then $a \cdot b < 0$.*
- *If $a < 0$ and $b > 0$, then $a \cdot b < 0$.*
- *If $a > 0$, then $a^{-1} > 0$.*
- *If $a < 0$, then $a^{-1} < 0$.*

Definition. The **absolute value** of a is

$$|a| := \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Theorem (Properties of the absolute value)**.**

- *Non-negativity:* $|x| \geq 0$.
- *Non-degeneracy:* $|x| = 0$ *if and only if* $x = 0$.
- *Symmetry:* $|-x| = |x|$.
- *The triangle inequality:* $|x + y| \leq |x| + |y|$.

Definition. The **distance** from a to b is

$$\operatorname{dist}(a, b) := |b - a|.$$

Theorem (Properties of the distance)**.**

- *Non-negativity:* $\operatorname{dist}(a, b) \geq 0$.
- *Non-degeneracy:* $\operatorname{dist}(a, b) = 0$ *if and only if* $a = b$.
- *Symmetry:* $\operatorname{dist}(a, b) = \operatorname{dist}(b, a)$.
- *The triangle inequality:* $\operatorname{dist}(a, c) \leq \operatorname{dist}(a, b) + \operatorname{dist}(b, c)$.

Postulate (P13), the “least upper bound property” (or “completeness”) of the real numbers, expresses the intuitive fact that “the real number line has no holes”. In Spivak’s book it appears in Chapter 8.

Let X be a set of real numbers. We recall and introduce some vocabulary:

- An **upper bound** for X is a number u such that $x \le u$ for all $x \in X$.
- X is **bounded from above** if it has an upper bound.
- X is **bounded** if it is bounded from above and from below.
- The **least upper bound** of X, also called the **supremum** of X and denoted $\sup X$, is a number u such that
  - u is an upper bound for X.
  - For every upper bound y for X, $u \le y$.

The least upper bound property of the real numbers, which we take as an axiom, is this:

(P13) Let $X \subseteq \mathbb{R}$ be a set of real numbers that is non-empty and bounded from above. Then X has a least upper bound.

We leave the proof of the following lemma as an exercise to the reader.

Lemma. *Let $X \subseteq \mathbb{R}$. Let u and $\widehat{u}$ be least upper bounds of X. Then $u = \widehat{u}$.*

We have the analogous notion of **lower bound**, **bounded from below**, and **greatest lower bound**, also called the **infimum** of X and denoted $\inf X$. For each of these notions, please write down its definition, please give one example and one non-example, and please determine if this notion is a number (which is a noun) or a property of the set X (which is an adjective describing X). Can you state a lemma that justifies calling this “*the* infimum” rather than “*an* infimum”?

The following theorem is a consequence of the least upper bound axiom. Can you prove it? (Hint: define $-X = \{-x \mid x \in X\}$. Then ℓ is a lower bound for X if and only if $-\ell$ is an upper bound for $-X$.)

Theorem. *Let $X \subseteq \mathbb{R}$ be a set of real numbers that is non-empty and bounded from below. Then X has a greatest lower bound.*