LIMITS AND CONTINUITY

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These notes supplement Chapters 5 and 6 of Spivak. Please let me know if you find a mistake or if any part of the notes is unclear.

Definition. Let $a \in \mathbb{R}$ and $\delta > 0$. The **δ-neighbourhood** of a is $(a-\delta, a+\delta)$. The **punctured**[1] **δ-neighbourhood** of a is $(a-\delta, a+\delta) \smallsetminus \{a\}$ (namely, $\{x \mid 0 < |x-a| < \delta\}$).

Notation: we can replace δ by other symbols, for example, $\delta_1,\ \delta_2,\ \epsilon,\ \widehat{\epsilon},\ \epsilon_1,\ \epsilon_2$.

Exercise (Exercise about neighbourhoods)**.**

(i) For any number ℓ, if ℓ is positive, then there exists an $\epsilon > 0$ such that all the numbers in the ϵ-neighbourhood of ℓ are positive.

(ii) For any two numbers a and b, if $a \neq b$, then there exists an $\epsilon > 0$ such that the ϵ-neighbourhood of a is disjoint from the ϵ-neighbourhood of b.

(iii) For any two numbers a and b, if $a < b$, then there exists a $\delta > 0$ such that, for every x_1 and x_2, if x_1 is in the δ-neighbourhood of a and x_2 is in the δ-neighbourhood of b, then $x_1 < x_2$.

(iv) For any number a, and for any $\delta_1 > 0$ and $\delta_2 > 0$, there exists a number x that is both in the δ_1-punctured neighbourhood of a and in the δ_2-punctured neighbourhood of a.

Definition. Fix a function f, a number a, and a number ℓ. We say that **$\mathbf{f(x)}$ approaches the limit ℓ as $\mathbf{x}$ approaches** a, and we write $\lim_{x\to a} f(x) = \ell$ or $f(x) \xrightarrow[x\to a]{} \ell$, if the following condition holds:

> For every $\epsilon > 0$ there exists $\delta > 0$ such that, for every x,
> if x is in the punctured δ-neighbourhood of a,
> then $f(x)$ is in the ϵ-neighbourhood of ℓ.

Equivalently,

> For every $\epsilon > 0$ there exists $\delta > 0$ such that, for every x,
> if $0 < |x-a| < \delta$, then $|f(x) - \ell| < \epsilon$.

(The point a need not be in the domain of f.)

Remarks: (1) Informally, the above condition means that we can guarantee that $f(x)$ be in the ϵ-neighbourhood of ℓ by requiring that x be in the punctured δ-neighbourhood of a. (2) We follow the conventions of Spivak's textbook, by which "$|f(x)-\ell| < \epsilon$" means that x is in the domain of f *and* the inequality holds.

[1] Some authors call this the "*deleted* δ-neighbourhood of a"

Assume that f is is defined on some punctured neighbourhood of a. The *negation* of $\lim_{x\to a} f(x) = \ell$ is this:

> There exists $\epsilon > 0$ such that, for every $\delta > 0$, there exists x such that
> $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \epsilon$

Example. The Heaviside function, given by $f(x) = 1$ when $x \geq 0$ and $f(x) = 0$ when $x < 0$, does not approach any limit at the point $x = 0$. We showed this in class in detail.

Example. Consider the identity function, $f(x) := x$. Then, for any a, we have $\lim_{x\to a} f(x) = a$.

Proof. Let $\epsilon > 0$. Let $\delta = \epsilon$. Let x be such that $0 < |x - a| < \delta$. Then

$$\begin{aligned} |f(x) - a| &= |x - a| && \text{because } f(x) = x \\ &< \delta && \text{by the assumption on } x \\ &= \epsilon && \text{by the choice of } \delta. \end{aligned}$$

□

Example. Let $c \in \mathbb{R}$. Let $f(x)$ be the constant function with value c. Then $\lim_{x\to a} f(x) = c$.

Proof. Let $\epsilon > 0$. Let $\delta = 1$. Let x be such that $0 < |x-a| < \delta$. Then $|f(x)-c| = 0 < \epsilon$. □

Lemma. *Suppose that $f(x) \xrightarrow[x\to a]{} \ell$ and that ℓ is positive. Then there exists $\delta > 0$ such that, for all x, if x is in the punctured δ-neighbourhood of a, then $f(x)$ is positive.*

Proof. Let $\epsilon > 0$ be such that all the numbers in the ϵ-neighbourhood of ℓ are positive. (Such an ϵ exists by the above exercise about neighbourhoods.)

Let $\delta > 0$ be such that, for all x, if x is in the punctured δ-neighbourhood of a, then $f(x)$ is in the ϵ-neighbourhood of ℓ. (Such a δ exists because $f(x) \xrightarrow[x\to a]{} \ell$.)

Let x be in the punctured δ-neighbourhood of a.
Then $f(x)$ is in the ϵ-neighbourhood of ℓ, by our choice of δ.
And so $f(x)$ is positive, by our choice of ϵ. □

Theorem (Uniqueness of limit)**.** *Suppose that $f(x) \xrightarrow[x\to a]{} \ell_1$ and $f(x) \xrightarrow[x\to a]{} \ell_2$. Then $\ell_1 = \ell_2$.*

Proof. Seeking a contradiction, suppose that $\ell_1 \neq \ell_2$. Let $\epsilon > 0$ be such that the ϵ-neighbourhood of ℓ_1 is disjoint from the ϵ-neighbourhood of ℓ_2. (Such an ϵ exists by the above exercise about neighbourhoods.)

Let $\delta_1 > 0$ be such that, for all x, if x is in the punctured δ_1-neighbourhood of a, then $f(x)$ is in the ϵ-neighbourhood of ℓ_1. (Such a δ_1 exists because $f(x) \xrightarrow[x\to a]{} \ell_1$.)

Let $\delta_2 > 0$ be such that, for all x, if x is in the punctured δ_2-neighbourhood of a, then $f(x)$ is in the ϵ-neighbourhood of ℓ_2. (Such a δ_2 exists because $f(x) \xrightarrow[x\to a]{} \ell_2$.)

Let x be a number that is both in the punctured δ_1-neighbourhood of a and in the punctured δ_2-neighbourhood of a. (Such an x exists by the above exercise about neighbourhoods.)

Then $f(x)$ is both in the ϵ-neighbourhood of ℓ_1 and in the ϵ-neighbourhood of ℓ_2, by our choice of δ_1 and of δ_2.

So the ϵ-neighbourhood of ℓ_1 is not disjoint from the ϵ-neighbourhood of ℓ_2, which contradicts our choice of ϵ. □

Definition. A function f is **continuous** at a point a if

$$\lim_{x\to a} f(x) = f(a).$$

(In particular, a must be in the domain of f.)

Exercise. a function f is continuous at a point a if and only if it satisfies the following condition.

> For every $\epsilon > 0$ there exists $\delta > 0$ such that, for every x, if $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

Exercise. Given a function $f\colon D \to \mathbb{R}$ and a point a,

$$\lim_{x\to a} f(x) = \ell$$

if and only if the function

$$g(x) := \begin{cases} f(x) & \text{if } x \neq a; \\ \ell & \text{if } x = a \end{cases}$$

is continuous at the point a.

Example. Consider the multiplicative inverse function $x \mapsto 1/x$ on $\mathbb{R} \setminus \{0\}$. According to the handout "Approximations", at any point $a \neq 0$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x-a| < \delta$ implies $|1/x - 1/a| < \epsilon$. Thus, the function $x \mapsto 1/x$ is continuous at a.

Example. Consider the square-root function $x \mapsto \sqrt{x}$ on $\mathbb{R}_{>0}$. According to the handout "Approximations", at any point $a > 0$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x-a| < \delta$ implies $|\sqrt{x} - \sqrt{a}| < \epsilon$. Thus, the function $x \mapsto \sqrt{x}$ is continuous at a.

Theorem. *Suppose that $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$. Then the following hold.*

(1) If $f(x) \geq g(x)$ for all x, then $\ell \geq m$.
(2) $\lim_{x\to a}(f+g)(x) = \ell + m$.
(3) $\lim_{x\to a}(f\cdot g)(x) = \ell \cdot m$.
(4) If $\ell \neq 0$, then $\lim_{x\to a} \dfrac{1}{f(x)} = \dfrac{1}{\ell}$;

Proof of Part (1) of the theorem.

Seeking a contradiction, assume that $\ell < m$

Let $\epsilon > 0$ be such that, for every t_1 and t_2, if t_1 is in the ϵ-neighbourhood of ℓ and t_2 is in the ϵ-neighbourhood of m, then $t_1 < t_2$. (Such an ϵ exists by the above exercise about neighbourhoods.)

Let $\delta_1 > 0$ be such that, for every x, if x is in the punctured δ_1-neighbourhood of a, then $f(x)$ is in the ϵ-neighbourhood of ℓ. (Such a δ_1 exists because $f(x) \xrightarrow[x\to a]{} \ell$.)

Let $\delta_2 > 0$ be such that, for every x, if x is in the punctured δ_2-neighbourhood of a, then $g(x)$ is in the ϵ-neighbourhood of m. (Such a δ_1 exists because $g(x) \xrightarrow[x\to a]{} m$.)

Let x be a number that is both in the δ_1-punctured neighbourhood of a and in the δ_2-punctured neighbourhood of a. (Such an x exists by the above exercise about neighbourhoods.)

Then $f(x)$ is in the ϵ-neighbourhood of ℓ and $g(x)$ is in the ϵ-neighbourhood of m (by our choices of δ_1 and δ_2).

So $f(x) < g(x)$ (by our choice of ϵ), which contradicts the assumption that $f(x) \geq g(x)$ for all x. $\square$

Proof of Part (2) of the theorem.
Let $\epsilon > 0$.
We need to find $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|(f + g)(x) - (\ell + m)| < \epsilon$.

As shown in the handout "Approximations" (in slightly different notation), there exists $\widehat{\epsilon} > 0$ such that if $|y_1 - \ell| < \widehat{\epsilon}$ and $|y_2 - m| < \widehat{\epsilon}$ then $|(y_1 + y_2) - (\ell + m)| < \epsilon$. Choose such an $\widehat{\epsilon}$.
Let $\delta_1 > 0$ be such that $0 < |x - a| < \delta_1$ implies $|f(x) - \ell| < \widehat{\epsilon}$.
Let $\delta_2 > 0$ be such that $0 < |x - a| < \delta_2$ implies $|g(x) - m| < \widehat{\epsilon}$.
We will show that $\delta := \min\{\delta_1, \delta_2\}$ is as required.

Let x be such that $0 < |x - a| < \delta$.
By our choice of δ, $|f(x) - \ell| < \widehat{\epsilon}$ and $|g(x) - m| < \widehat{\epsilon}$.
By our choice of $\widehat{\epsilon}$, $|(f(x) + g(x)) - (\ell + m)| < \epsilon$, as required. $\square$

Proof of Part (3) of the theorem.
Let $\epsilon > 0$.
We need to find $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|(fg)(x) - (\ell m)| < \epsilon$.

As shown in the handout "Approximations" (in slightly different notation), there exists $\widehat{\epsilon} > 0$ such that if $|y_1 - \ell| < \widehat{\epsilon}$ and $|y_2 - m| < \widehat{\epsilon}$ then $|(y_1 y_2) - (\ell m)| < \epsilon$. Choose such an $\widehat{\epsilon}$.
Let $\delta_1 > 0$ be such that $0 < |x - a| < \delta_1$ implies $|f(x) - \ell| < \widehat{\epsilon}$.
Let $\delta_2 > 0$ be such that $0 < |x - a| < \delta_2$ implies $|g(x) - m| < \widehat{\epsilon}$.
We will show that $\delta := \min\{\delta_1, \delta_2\}$ is as required.

Let x be such that $0 < |x - a| < \delta$.
By our choice of δ, $|f(x) - \ell| < \widehat{\epsilon}$ and $|g(x) - m| < \widehat{\epsilon}$.
By our choice of $\widehat{\epsilon}$, $|f(x)g(x) - (\ell m)| < \epsilon$, as required. $\square$

Proof of Part (4) of the theorem.
Let $\epsilon > 0$.
We need to find $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|\frac{1}{f(x)} - \frac{1}{\ell}| < \epsilon$.

As shown in the handout "Approximations" (in slightly different notation), there exists $\widehat{\epsilon} > 0$ such that if $|y - \ell| < \widehat{\epsilon}$ then $|\frac{1}{y} - \frac{1}{\ell}| < \epsilon$. Choose such an $\widehat{\epsilon}$.
Let $\delta > 0$ be such that $0 < |x - a| < \delta$ implies $|f(x) - \ell| < \widehat{\epsilon}$.
We will show that this δ is as required.

Let x be such that $0 < |x - a| < \delta$.
By our choice of δ, $|f(x) - \ell| < \widehat{\epsilon}$.
By our choice of $\widehat{\epsilon}$, $|\frac{1}{f(x)} - \frac{1}{\ell}| < \epsilon$, as required. $\square$

Theorem. *Suppose that* $\lim_{x\to a} f(x) = b$*, that* $\lim_{t\to b} h(t) = \ell$*, and that* h *is continuous at the point* b*. Then* $\lim_{x\to a} h(f(x)) = \ell$*.*

Proof. Let $\epsilon > 0$.
We need to find $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|h(f(x)) - \ell| < \epsilon$.

Let $\widehat{\epsilon} > 0$ be such that $0 < |t-b| < \widehat{\epsilon}$ implies $|h(t) - \ell| < \epsilon$.
Let $\delta > 0$ be such that $0 < |x-a| < \delta$ implies $|f(x) - b| < \widehat{\epsilon}$.
We will show that this δ is as required.

Let x be such that $0 < |x-a| < \delta$.
By our choice of δ, $|f(x) - b| < \widehat{\epsilon}$.
We now consider two cases.

Case 1: $f(x) \neq b$.
Then $0 < |f(x) - b| < \widehat{\epsilon}$, and by our choice of $\widehat{\epsilon}$, $|h(f(x)) - \ell| < \epsilon$, as required.

Case 2: $f(x) = b$.
Then $h(f(x)) = h(b)$. Because h is continuous at b, and by the uniqueness of the limit, $h(b) = \ell$. So $h(f(x)) = \ell$, and so $|h(f(x)) - \ell| < \epsilon$.

In each of these two cases, we obtain $|h(f(x)) - \ell| < \epsilon$, as required. □

Exercise. Suppose that $\lim_{x\to a} f(x) = b$ and that $\lim_{t\to b} h(t) = \ell$. Can we still conclude that $\lim_{x\to a} h(f(x)) = \ell$? If we can, prove it. If not, find a counterexample.