APPLICATIONS OF DERIVATIVES.

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These notes supplement Chapter 11 of Spivak. Please let me know if you find a mistake or if any part of the notes is unclear.

Please make sure that you are comfortable with the following differentiation rules and with their proofs.

Additivity: If f and g are differentiable at a then so is $f+g$, and $(f+g)'(a) = f'(a)+g'(a)$.

Leibnitz rule (= **product rule**): If f and g are differentiable at a, then so is $f \cdot g$, and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

Chain rule: If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

Please make sure that you are comfortable with the following examples of derivatives and with their proofs.

$f(x)$	$f'(x)$
constant	0
x	1
x^n	nx^{n-1}
$x^{1/2}$	$\frac{1}{2}x^{-1/2}$
x^{-1}	$-x^{-2}$

The chain rule and $(\frac{1}{x})' = -\frac{1}{x^2}$ together imply that $(\frac{1}{g})' = \frac{-g'}{g^2}$. Please make sure that you see how. This and the product rule further imply that $(\frac{f}{g})' = \frac{f' \cdot g - f \cdot g'}{g^2}$. Please make sure that you see how.

Please memorize the following examples of derivatives.

$f(x)$	$f'(x)$
$\exp(x)$	$\exp(x)$
$\log(x)$	$1/x$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

We will define these functions rigorously later in the year. Note that log denotes the natural (base e) logarithm; see the video "ln vs log x" here: `https://tinyurl.com/y2gw3xs7`

Please also make sure that you are able to derive the formulas for the derivatives of the other trigonometric functions from the differentiation rules and from the formulas for the derivatives of the sine and cosine functions.

Definition. Suppose that f is defined near γ (i.e. on some neighbourhood of γ). Then γ is a **local maximum point** for f if there exists a $\delta > 0$ such that $f(\gamma) \geq f(x)$ for all $x \in (\gamma - \delta, \gamma + \delta)$, and γ is a **strict local maximum point** for f if there exists a $\delta > 0$ such that $f(\gamma) > f(x)$ for all $x \in (\gamma - \delta, \gamma + \delta) \setminus \{\gamma\}$.

Warning: in this definition we do not consider the case that f is defined on a closed interval $[a, b]$ and γ is an endpoint of this interval. In this case, some authors (including Spivak) still call a a local maximum point if the restriction of f to some right neighbourhood of a takes its maximum at the point a, and similarly for b. If you run into a usage of "local maximum" where you're not sure if this term allows an endpoint of an interval, please ask.

Exercise: Give an analogous definition of a "local minimum point". Say what it means for 1 to be a local minimum point for a function $g\colon [0, 3] \to \mathbb{R}$. Show that, for any function $f\colon [a, b] \to \mathbb{R}$, if $\gamma \in (a, b)$ is a minimum point for f, then γ is also a local minimum point for f.

Theorem (Fermat)**.** *Suppose that $x \in (a, b)$ is a local maximum point or a local minimum point for a function $f\colon (a, b) \to \mathbb{R}$ and that f is differentiable at x. Then $f'(x) = 0$.*

Proof. See your class notes or Spivak's book. $\square$

Corollary. *If $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and if $x \in [a, b]$ is a maximum point or a minimum point for f, then $x = a$ or $x = b$ or $f'(x) = 0$.*

Definition. Suppose that a function f is differentiable at a point x. The point x is a **critical point** (**=stationary point**) if $f'(x) = 0$.

Warning: in this definition we do not consider the case that f is not differentiable at x. In this case, some authors (not including Spivak) still call x a critical point of f. If you run into a usage of "critical point" where you're not sure if this term allows a point where the function is not differentiable, please ask.

Example. 0 is a stationary point for $x \mapsto x^3$ (but not a local minimum nor local maximum).

Theorem (**Rolle's Theorem**)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = f(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = 0$.*

Sketch of proof. By the extreme value theorem for continuous functions, the function f attains a minimum value and a maximum value on the interval $[a, b]$. If the minimum and maximum of f are both attained at the endpoints of the interval, then f is constant, and any x will do. Otherwise, take x to be a minimum point or maximum point for f that is not an endpoint of $[a, b]$ and apply the above corollary. $\square$

Theorem (**Mean Value Theorem**)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.*

Sketch of proof. Subtract a linear function with slope $\frac{f(b)-f(a)}{b-a}$. This results in a new function that takes the same values at a and at b. Apply Rolle's theorem to it. $\square$

Corollary. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant.*

Proof. Let c and d be two distinct points in $[a,b]$. Suppose that $c < d$ (otherwise rename them). By the mean value theorem for f on $[c,d]$, there exists $x \in (c,d)$ such that $\frac{f(d)-f(c)}{d-c} = f'(x)$. Because the right hand side is zero, $f(c) = f(d)$. Because c and d were arbitrary, f is constant. $\square$

Definition. A function f is

(strictly) **increasing** if $x < y$ implies $f(x) < f(y)$.
(strictly) **decreasing** if $x < y$ implies $f(x) > f(y)$.
weakly increasing (equivalently, **non-decreasing**) if $x < y$ implies $f(x) \leq f(y)$.
weakly decreasing (equivalently, **non-increasing**) if $x < y$ implies $f(x) \geq f(y)$.

Corollary of the mean value theorem.
Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

If $f'(t) = 0$ for all $t \in (a,b)$, then f is constant on $[a,b]$.
If $f'(t) > 0$ for all $t \in (a,b)$, then f is increasing on $[a,b]$.
If $f'(t) < 0$ for all $t \in (a,b)$, then f is decreasing on $[a,b]$.
If $f'(t) \geq 0$ for all $t \in (a,b)$, then f is weakly increasing on $[a,b]$.
If $f'(t) \leq 0$ for all $t \in (a,b)$, then f is weakly decreasing on $[a,b]$.

Proof of the last item. Assume that $f' \leq 0$ on (a,b). Let x, y be any two points in $[a,b]$ such that $x < y$. By the mean value theorem for f on $[x,y]$, there exists $t \in (x,y)$ such that $f'(t) = \frac{f(y)-f(x)}{y-x}$. Since the left hand side is $\leq$ and the denominator of the right hand side is > 0, the numerator of the right hand side is ≤ 0. This implies that $f(x) \geq f(y)$. $\square$

Exercise. Prove the other items.

Recall that $f''(a) = \lim\limits_{x \to a} \frac{f'(x)-f'(a)}{x-a}$.

Note: $\quad f$ is twice differentiable at a

$\implies$ f is differentiable near a and f' is (differentiable, hence) continuous at a

$\implies$ f is differentiable at a .

Theorem. *Suppose that $f''(a)$ exists. (In particular, f is defined and is differentiable near a, i.e., on some neighbourhood of a.)*
If $f'(a) = 0$ and $f''(a) < 0$, then f has a strict local maximum at a.
If $f'(a) = 0$ and $f''(a) > 0$, then f has a strict local minimum at a.

Proof. We prove the first statement, leaving the second statement as an exercise.

Assume $f'(a) = 0$ and $f''(a) < 0$.

The assumption $\lim_{x\to a} \frac{f'(x)-f'(a)}{x-a} < 0$ implies that there exists $\delta > 0$ such that $\frac{f'(\overline{x})-f'(a)}{\overline{x}-a} < 0$ for all $\overline{x}$ in the punctured δ-neighbourhood of a. Since $f'(a) = 0$, this means that $f'(\overline{x}) < 0$ if $a < \overline{x} < a + \delta$ and $f'(\overline{x}) > 0$ if $a - \delta < \overline{x} < a$.

Because f is twice differentiable at a, f is (differentiable, hence) continuous near a, so there exists $\delta > 0$ such that f is continuous on $[a - \delta, a + \delta]$.

Fix a δ that satisfies both of these conditions.

Since f is continuous on $[a, a + \delta]$ and $f' < 0$ on $(a, a + \delta)$, f is decreasing on $[a, a + \delta]$. Since f is continuous on $[a - \delta, a]$ and $f' > 0$ on $(a - \delta, a)$, f is increasing on $[a - \delta, a]$.

In particular, if $a < \overline{x} < a + \delta$ then $f(a) > f(\overline{x})$ and if $a - \delta < \overline{x} < a$ then $f(\overline{x}) < f(a)$.

It follows that $f(\overline{x}) < f(a)$ for all $\overline{x} \in (a - \delta, a + \delta) \setminus \{a\}$, as required. $\square$

Corollary. *Suppose that $f''(a)$ exists.*
If a is a local minimum point for f, then $f''(a) \geq 0$.
If a is a local maximum point for f, then $f''(a) \leq 0$.

Proof. Seeking a contradiction, suppose that a is a local minimum point for f (which implies that $f'(a) = 0$) and that $f''(a) < 0$. By the theorem, there exists $\delta_1 > 0$ such that $f(x) < f(a)$ for all x in the punctured δ_1-neighbourhood of a. Because a is a local minimum, there exists $\delta_2 > 0$ such that $f(x) \geq f(a)$ for all x in the δ_2-neighbourhood of a. Because the punctured δ_1-neighbourhood is not disjoint from the δ_2-neighbourhood (for example, they both contain $x := a + \frac{1}{2}\min\{\delta_1, \delta_2\}$), we obtain a contradiction. This proves the first statement. We leave the second statement as an exercise for the reader. $\square$

Theorem. *Suppose that f is continuous at a, that f is differentiable on some punctured neighbourhood of a, and that $\lim_{x\to a} f'(x) = c$. Then f is differentiable at a, and $f'(a) = c$.*

Exercise. The function that is defined as $x\cos(1/x)$ for $x \neq 0$ and as 0 for $x = 0$ is continuous everywhere and differentiable when $x \neq 0$ but is not differentiable at $x = 0$. Why does it not violate the statement of the theorem?

Proof of the theorem. Let $\epsilon > 0$. Let $\delta > 0$ be such that $0 < |\overline{x} - a| < \delta$ implies that $|f'(\overline{x}) - c| < \epsilon$. (Why does such a δ exist?)

Let x be such that $0 < |x - a| < \delta$.

Case 1: $a < x < a + \delta$. Since f is continuous on $[a, x]$ and differentiable on (a, x), by the mean value theorem there exists $\overline{x}$ in (a, x) such that $\frac{f(x)-f(a)}{x-a} = f'(\overline{x})$. We have

$$\left|\frac{f(x) - f(a)}{x - a} - c\right| = |f'(\overline{x}) - c| < \epsilon,$$

where the first equality is by the choice of $\overline{x}$ and the second is since ($a < \overline{x} < x < a + \delta$, hence) $0 < |\overline{x} - a| < \delta$ and by the choice of δ.

Case 2: $a - \delta < x < a$. A similar argument shows that $\left|\frac{f(x)-f(a)}{x-a} - c\right| < \epsilon$ in this case too.

Putting together the two cases, we find that $0 < |x - a| < \delta$ implies that $\left|\frac{f(x)-f(a)}{x-a} - c\right| < \epsilon$, as required. $\square$