DIFFERENTIATION.

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These notes supplement Chapters 9, 10 of Spivak.

Definition. A function f is **differentiable** at a point a if (it is defined near[1] a and) the following limit exists:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This limit is called the **derivative** of f at a. It is denoted $f'(a)$ or $\left.\frac{df}{dx}\right|_{x=a}$.

Equivalently: it is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

The difference quotient $\frac{f(x) - f(a)}{x - a}$ is the slope of the "secant line" to the graph of f that passes through the points $(a, f(a))$ and $(x, f(x))$ on the graph.

Recall: The linear function that takes a to b and has slope c is $x \mapsto b + c(x - a)$. Its graph is given by the equation $y = b + c(x - a)$.

If f is differentiable at a, the **linear approximation** of f near a is the linear function L that takes a to $f(a)$ and has slope $f'(a)$, and the **tangent line** to the graph of f is the graph of L. Thus, this linear approximation is $L(x) = f(a) + f'(a)(x - a)$, and this tangent line is given by the equation $y = f(a) + f'(a)(x - a)$.

Lemma (Carathéodory criterion). *Let $f(x)$ be a function that is defined near a. Then*

(i) $f(x)$ is differentiable at $x = a$.

if and only if

(ii) There exists a function $h(x)$ that is continuous at a and such that, for all x, we have $f(x) = f(a) + (x - a)h(x)$.

Moreover, if (i) and (ii) hold, then $h(a) = f'(a)$.

Proof. Suppose that f is differentiable at a. Let $c := f'(a)$. Define

$$h(x) := \begin{cases} \dfrac{f(x) - f(a)}{x - a} & x \neq a \\ c & x = a. \end{cases}$$

[1] "defined near a" means that it is defined on some neighbourhood of a: there exists a neighbourhood $(a - \delta, a + \delta)$ of a where the function is defined.

Then $h(x)$ is continuous at a (because, by assumption, $\frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} c$) and, for all x, we have $f(x) = f(a) + (x-a)h(x)$ (which follows from the definition of h).

Conversely, suppose that $h(x)$ is continuous at a and that $f(x) = f(a) + (x-a)h(x)$ for all x. This equation implies that $\frac{f(x)-f(a)}{x-a} = h(x)$ whenever $x \neq a$. By continuity, $h(x) \xrightarrow[x\to a]{} h(a)$. So $\frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} h(a)$, which means that f is differentiable at a and $f'(a) = h(a)$. $\square$

Theorem. *If f is differentiable at a, then f is continuous at a.*

Proof. By Carathéodory's criterion, we can write

$$f(x) = f(a) + (x-a)h(x)$$

where h is continuous at a. By the arithmetic properties of limits, as x goes to a, the right hand side converges to $f(a)$. $\square$

The composition of a linear function with slope c and a linear function with slope $\widetilde{c}$ is a linear function with slope $\widetilde{c}c$ (please make sure that you see why!). Similarly, the composition of two differentiable functions is differentiable, and the derivative of the composition is the product of the derivatives, at the relevant points:

Theorem (Chain rule). *Let $g(x)$ be differentiable at a, and let $f(y)$ be differentiable at $g(a)$. Then $(f \circ g)(x)$ is differentiable at a, and $(f \circ g)'(a) = f'(g(a))g'(a)$.*

Proof. Applying Carathéodory's criterion to g at the point a and to f at the point $b := g(a)$, we get

$$g(x) = g(a) + (x-a)h(x) \quad \text{and} \quad f(y) = f(b) + (y-b)\widetilde{h}(y),$$

where $h(x)$ is continuous at a and $h(a) = g'(a)$, and where $\widetilde{h}(y)$ is continuous at b and $\widetilde{h}(b) = f'(b)$. We compose:

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f(b) + (y-b)\,\widetilde{h}(y) \qquad \text{where } y = g(x) = b + (x-a)\,h(x) \\ &= f(b) + (x-a)\,h(x)\,\widetilde{h}(y) \qquad \text{since } y - b = (x-a)h(x), \\ &= (f \circ g)(a) + (x-a)H(x),\end{aligned}$$

where $H(x) = h(x)\,\widetilde{h}(y) = h(x)\,\widetilde{h}(b + (x-a)h(x))$. Because $b + (x-a)h(x)$ is continuous at the point a and takes the value b at that point, and because $\widetilde{h}(y)$ is continuous at the point b, their composition $\widetilde{h}(b + (x-a)h(x))$ is continuous at the point a, and so is its product with $h(x)$ (because $h(x)$ is continuous at the point a). So $H(x)$ is continuous at the point a. By Carathéodory's criterion for $f \circ g$, we conclude that the derivative of $(f \circ g)(x)$ at the point a exists and is equal to $H(a)$, which, in turn, is equal to $\widetilde{h}(b)h(a)$, which is $f'(b)g'(a)$. $\square$