“HARD THEOREMS” ABOUT CONTINUOUS FUNCTIONS

MAT157, FALL 2020. YAEL KARSHON.

These notes supplement Chapters 7 and 8 of Spivak. They are about properties of continuous functions that follow from the least upper bound property of the real numbers. Please let me know if you find a mistake or if any part of the notes is unclear.

Before reading this note, please review the definitions and basic properties of least upper bounds and greatest lower bounds.

We already defined what it means for a function to be continuous at a point, or right-continuous or left-continuous at a point. We say that a function $f\colon [a,b] \to \mathbb{R}$ is **continuous on the closed interval** $[a,b]$ if it is continuous at every point of the open interval (a,b), right-continuous at a, and left-continuous at b.

Lemma (**Sign preserving property of continuous functions**). *Suppose that a function f is continuous from the left at a point γ.*

- *If $f(\gamma) > 0$, then there is some left neighbourhood of γ on which $f > 0$.*
- *If $f(\gamma) < 0$, then there is some left neighbourhood of γ on which $f < 0$.*

Consequently,

- *If there is $\delta > 0$ such that $f > 0$ on $(\gamma - \delta, \gamma)$, then $f(\gamma) \geq 0$.*
- *If there is $\delta > 0$ such that $f < 0$ on $(\gamma - \delta, \gamma)$, then $f(\gamma) \leq 0$.*

Exercise.

(1) State an analogous lemma for a function that is continuous from the right.
(2) In the above lemma, deduce the third statement from the second statement, and deduce the fourth statement from the first statement.
(3) In the above lemma, prove the first statement.

Now comes our first “hard theorem”.

Theorem (**Bolzano’s theorem**). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. Suppose that $f(a) < 0 < f(b)$ or that $f(a) > 0 > f(b)$. Then there exists a γ such that $a < \gamma < b$ and $f(\gamma) = 0$.*

(Note that the analogous theorem for rational numbers is false. For example, $x \mapsto x^2 - 3$ is a continuous function that takes rational numbers to rational numbers, its value at $x = 0$ is negative and its value at $x = 10$ is positive, but there is no rational number between 0 and 10 where the value of this function is zero.)

Proof of Bolzano’s theorem. Without loss of generality, assume that $f(a) < 0 < f(b)$. The other case is similar.

We take the convention $[a, a] = \{a\}$. Let

$$A = \{x \in [a, b] \mid f < 0 \text{ on } [a, x]\}.$$

Step 1: The set A contains some number that is greater than a and is bounded from above by some number that is lesser than b.

Indeed, by the sign preserving property of continuous functions, there exist some right neighbourhood $[a, a + \delta_1)$ of a where $f < 0$ and left neighbourhood $(b - \delta_2, b]$ of b where $b > 0$. Then A contains $a + \delta_1/2$ and (is disjoint from $(b - \delta_2, b]$, hence) is bounded from above by $b - \delta_2$.

Step 2: The supremum $\gamma := \sup A$ exists, and it satisfies $a < \gamma < b$.

Indeed, by Step 1 and the least upper bound axiom, the set A has a supremum, which is greater than a (because it's an upper bound for a set that contains a number greater than a) and lesser than b (because it's a least upper bound for a set that has an upper bound that is lesser than b).

Step 3: $f < 0$ on $[a, \gamma)$.

Indeed, let $x \in [a, \gamma)$. Because x is lesser than the least upper bound γ, it itself is not an upper bound. So there is $y \in A$ such that $y > x$. Fix such a y. Because $a \leq x < y$, we have $x \in [a, y]$. Because $y \in A$, we have $f < 0$ on $[a, y]$. But $x \in [a, y]$, so $f(x) < 0$.

Step 4: $f(\gamma) \leq 0$.

Indeed: by Step 3, $f < 0$ on $[a, \gamma)$. Because f is continuous from the left at γ, and by the corollary to the sign preserving property of continuous functions, $f(\gamma) \leq 0$.

Step 5: $f(\gamma) \not< 0$.

Indeed, seeking a contradiction, assume that $f(\gamma) < 0$. By Step 2, $\gamma < b$, so f is continuous from the right at γ. By the sign preserving property of continuous functions, there exists $\delta > 0$ such that $f < 0$ on $[\gamma, \gamma + \delta)$. Pick such a δ. Since also $f < 0$ on $[a, \gamma)$ (by Step 3), we conclude that $f < 0$ on $[a, \gamma + \delta)$, and hence on $[a, \gamma + \frac{1}{2}\delta]$. So $\gamma + \frac{1}{2}\delta \in A$, which contradicts the fact that γ is an upper bound for A.

Conclusion: by Steps 2 and 4 and 5, we have that $a < \gamma < b$ and $f(\gamma) = 0$. □

Theorem (Intermediate value theorem for continuous functions). *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) \neq f(b)$. Let β be any number between $f(a)$ and $f(b)$. Then there exists a γ such that $a < \gamma < b$ and $f(\gamma) = \beta$.*

Proof. Apply Bolzano's theorem to the function $x \mapsto f(x) - \beta$. □

Here is some more vocabulary. Given a function $f\colon D \to \mathbb{R}$,

- An **upper bound** for f is a number u such that $f(x) \leq u$ for all $x \in D$.
- f is **bounded from above** if it has an upper bound.
- A **maximum value** for f is a number y such that
  $y = f(x)$ for some $x \in D$, and
  $y \geq f(x)$ for all $x \in D$.

Also,

- A **maximum point** for f is a point $x \in D$ such that $f(x) \geq f(x')$ for all $x' \in D$.

Exercise. Give analogous definitions for the notions **lower bound**, **bounded from below**, **minimum value**, and **minimum point**.

A function is **bounded** if it is bounded from above and from below.

Thus, an upper bound for f is the same thing as an upper bound for the set of values $\{f(x) \mid x \in D\}$, and a maximum for f is the same thing as a maximal element of this set of values.

Lemma (**Union of bounded functions**). *Let D_1 and D_2 be sets that are contained in the domain of definition of a function f. Suppose that f is bounded on each of D_1 and D_2. Then f is bounded on their union, $D_1 \cup D_2$.*

Lemma (**Continuous functions are locally bounded**).

(L) Suppose that a function f is continuous from the left at a point γ. Then there exists $\delta > 0$ such that f is bounded on $(\gamma - \delta, \gamma]$.
(R) Suppose that a function f is continuous from the right at a point γ. Then there exists $\delta > 0$ such that f is bounded on $[\gamma, \gamma + \delta)$.

Exercise. Prove these two lemmas.

Now comes our second "hard theorem".

Theorem (**Boundedness theorem for continuous functions**). *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. Then f is bounded.*

(Note that the analogous theorem for rational numbers is false. For example, $x \mapsto \dfrac{1}{x^2-3}$ is a continuous function that takes rational numbers to rational numbers, but it is not bounded from above on $\mathbb{Q} \cap [0, 10]$.)

Proof. Let
$$A = \{x \in [a,b] \mid f \text{ is bounded on } [a,x]\}.$$

Step 1: The supremum $\gamma := \sup A$ exists and satisfies $a \leq \gamma \leq b$.

Indeed, this is because the set A contains a and is bounded from above by b. (Exercise: complete the details of this argument.)

Step 2: $\gamma \in A$.

Indeed, we now show this. Because $\gamma \geq a$, we have that $\gamma = a$ or $\gamma > a$. In the case $\gamma = a$, we have already shown that $a \in A$. Suppose now that $\gamma > a$.

Then f is continuous from the left at γ. By Part (L) of the above Lemma, there is $\delta > 0$ such that f is bounded on $(\gamma - \delta, \gamma]$. Pick such a δ.

γ is a least upper bound for A, so $\gamma - \delta$ is not an upper bound for A, so there exists $x \in A$ such that $x > \gamma - \delta$. Pick such an x. Then the union $[a, x] \cup (\gamma - \delta, \gamma]$ contains $[a, \gamma]$ (moreover, the union is equal to $[a, \gamma]$).

Since $x \in A$, f is bounded on $[a, x]$. By the choice of δ, f is bounded on $(\gamma - \delta, \gamma]$. By the Lemma, f is bounded on the union $[a, x] \cup (\gamma - \delta, \gamma]$. So f is bounded on $[a, \gamma]$. This exactly means that $\gamma \in A$.

Step 3: $\gamma \not< b$.

Indeed, seeking a contradiction, assume that $\gamma < b$. Then f is continuous from the right at γ. By Part (R) of the above Lemma, there exists $\delta > 0$ such that f is bounded on $[\gamma, \gamma + \delta)$. Pick such a $\delta > 0$. Since f is also bounded on $[a, \gamma]$ (by Step 1), f is bounded on the union $[a, \gamma] \cup [\gamma, \gamma + \delta)$. Because $[a, \gamma + \frac{1}{2}\delta]$ is contained in this union, f is bounded on it. So $\gamma + \frac{1}{2}\delta \in A$, which contradicts the fact that γ is an upper bound for A.

Conclusion. By Step 1, $a \leq \gamma \leq b$. By step 3, $\gamma \not< b$. So $\gamma = b$. By Step 2, $\gamma \in A$. So $b \in A$. So f is bounded on $[a, b]$. □

And here is our third "hard theorem".

Theorem (Extreme value theorem for continuous functions). *Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function on a closed interval. Then f attains its extrema. I.E., there exist $y, z \in [a, b]$ such that $f(z) \leq f(x) \leq f(y)$ for all $x \in [a, b]$.*

(Note that the analogous theorem for rational numbers is false. For example, $x \mapsto 1-(x^2-3)^2$ is a continuous function that takes rational numbers to rational numbers, but it does not have a maximum value on $\mathbb{Q} \cap [0, 10]$.)

Proof. The set of values $\{f(x) \mid x \in [a, b]\}$ is non-empty. By the bounded theorem for continuous functions, it is bounded. By the least upper bound property of the real numbers, we can define

$$\beta := \sup\{f(x) \mid x \in [a, b]\}.$$

Then $f(x) \leq \beta$ for all $x \in [a, b]$, and we would like to show that there exists $y \in [a, b]$ such that $f(y) = \beta$.

Seeking a contradiction, suppose $f(x) \neq \beta$ for all $x \in [a, b]$. Then the function

$$\frac{1}{\beta - f(x)} \colon [a, b] \to \mathbb{R}$$

is well defined and continuous, and everywhere positive. By the boundedness theorem for continuous functions, this function has an upper bound; pick one, say, N. Then, for all $x \in [a, b]$, $0 < \dfrac{1}{\beta - f(x)} \leq N$, which implies that $f(x) \leq \beta - \frac{1}{N}$. So $\beta - \frac{1}{N}$ is an upper bound for the set of values of f, contradicting the fact that β is a *least* upper bound for this set. □