UNIFORM CONTINUITY.

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Uniform continuity is treated in the appendix to Spivak's Chapter 8. Spivak explains well the big picture. In this note we give a proof of the main theorem that emphasizes details.

Let $f\colon [a,b] \to \mathbb{R}$ be a function on an interval $[a,b]$.

f is continuous on $[a,b]$ if and only if for every $x \in [a,b]$ and every $\epsilon > 0$ there exists $\delta > 0$ such that for every $y \in [a,b]$, if $|y-x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

(Can you see why this is an easy consequence of the definition of continuity on an interval?)

f is **uniformly continuous** on $[a,b]$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in [a,b]$, if $|y-x| < \delta$ then $|f(y) - f(x)| < \epsilon$.

(Can you see how this is different from ordinary continuity?)

In this note we focus on closed intervals. But this definition of uniform continuity also applies to all intervals: (a,b), $[a,\infty)$, $\mathbb{R}$, etc.

Example. The function $x \mapsto \frac{1}{x}$ is not uniformly continuous on the interval $(0,1)$.

(Try to understand why this is true, informally. Then try to prove it, formally.)

Lemma. *Let $\epsilon > 0$.*

- *(L) Suppose that a function f is left-continuous at a point γ. Then there exists $\delta_1 > 0$ such that, for all y, z in $(\gamma - \delta_1, \gamma]$, we have $|f(y) - f(z)| < \epsilon$.*
- *(R) Suppose that a function f is right-continuous at a point γ. Then there exists $\delta_1 > 0$ such that, for all y, z in $[\gamma, \gamma + \delta_1)$, we have $|f(y) - f(z)| < \epsilon$.*
- *(LR) Suppose that a function f is continuous at a point γ. Then there exists $\delta_1 > 0$ such that, for all y, z in $(\gamma - \delta_1, \gamma + \delta_1)$, we have $|f(y) - f(z)| < \epsilon$.*

Exercise. Prove this lemma.

The following theorem is of a similar nature to what Spivak calls the "three hard theorems".

Theorem. *If $f\colon [a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$, then f is uniformly continuous on $[a,b]$.*

Proof. Fix $\epsilon > 0$. Let

$$A_\epsilon := \{\, x \in [a,b] \mid \text{ there exists } \delta > 0 \text{ such that for all } y, z \in [a,x], \\ \text{if } |y - z| < \delta, \text{ then } |f(y) - f(z)| < \epsilon \,\}.$$

Step 1. $a \in A_\epsilon$ (why?). Also, A_ϵ is bounded from above by b (why?). Let

$$\gamma = \sup A_\epsilon.$$

(Why does the supremum exist?)

We have $a \le \gamma \le b$. (Why is $\gamma \ge a$? Why is $\gamma \le b$?)

Step 2: $\gamma \in A_\epsilon$. Indeed:

Since $\gamma \ge a$ and $a \in A_\epsilon$, without loss of generality we may assume that $\gamma > a$. (Why?)

So f is left-continuous at γ. Let $\delta_1 > 0$ be as in Part (L) of the lemma.

Let $x \in A_\epsilon$ be such that $\gamma - \frac{\delta_1}{2} < x \le \gamma$.

(Why does there exist an x such that $x \in A_\epsilon$ and $x > \gamma - \frac{\delta_1}{2}$? Why is $x \le \gamma$?)

Let $\delta_2 > 0$ be such that, for every y and z in $[a, x]$, if $|y - z| < \delta_2$ then $|f(y) - f(z)| < \epsilon$.

(Why does such a δ_2 exist?)

Note that

$$[a, \gamma] = [a, \gamma - \frac{\delta_1}{2}] \cup [\gamma - \frac{\delta_1}{2}, \gamma].$$

Let $\delta = \min\{\frac{\delta_1}{2}, \delta_2\}$. Let $y, z \in [a, \gamma]$ be such that $|y - z| < \delta$.

Case 1: suppose that at least one of y or z is in $[\gamma - \frac{\delta_1}{2}, \gamma]$. Then y, z are both in $(\gamma - \delta_1, \gamma]$ (why?). So $|f(y) - f(z)| < \epsilon$ (by the choice of δ_1).

Case 2: suppose that y, z are both in $[a, \gamma - \frac{\delta_1}{2}]$. Then they are both in $[a, x]$ (why?). So $|f(y) - f(z)| < \epsilon$ (why?).

In either case, we get that $|f(y) - f(z)| < \epsilon$. Because y, z must satisfy the assumption of Case 1 or the assumption of Case 2 (why?), we get that $|f(y) - f(z)| < \epsilon$. Because $y, z \in [a, \gamma]$ were arbitrary and by the definition of A_ϵ, we conclude that $\gamma \in A_\epsilon$.

Step 3: $\gamma \not< b$. Indeed:

Seeking a contradiction, assume that $\gamma < b$. Let $\delta_1 > 0$ be as in the lemma. (If $a < \gamma < b$, use Part (LR) of the lemma; if $a = \gamma$, use Part (R) of the lemma.) Note that

$$[a, \gamma + \frac{\delta_1}{2}] = [a, \gamma] \cup [\gamma, \gamma + \frac{\delta_1}{2}].$$

By Step 2, we can choose $\delta_2 > 0$ such that, for all $y, z \in [a, \gamma]$, if $|y - z| < \delta_2$ then $|f(y) - f(z)| < \epsilon$. Fix such a δ_2. Let $\delta = \min\{\frac{\delta_1}{2}, \delta_2\}$. Note that $\delta > 0$.

Let $y, z \in [a, \gamma + \frac{\delta_1}{2}]$ be such that $|y - z| < \delta$.

If $\gamma = a$, then $y, z \in [\gamma, \gamma + \delta_1)$ (why?), and $|f(y) - f(z)| < \epsilon$ by the choice of δ_1. Otherwise, if at least one of y or z is in $[\gamma, \gamma + \frac{\delta_1}{2}]$, then because $|y - z| < \delta \le \frac{\delta_1}{2}$, both y and z are in $(\gamma - \delta_1, \gamma + \delta_1)$ (as a consequence of the triangle inequality), and so $|f(y) - f(z)| < \epsilon$ (by the choice of δ_1). If both y and z are in $[a, \gamma]$, then because $|y - z| < \delta_2$, we have $|f(y) - f(z)| < \epsilon$ (by the choice of δ_2).

In all cases, $|f(y) - f(z)| < \epsilon$. So we found a $\delta > 0$ such that, if $y, z \in [a, \gamma + \frac{\delta_1}{2}]$ and $|y - z| < \delta$, then $|f(y) - f(z)| < \epsilon$. The existence of such a δ means that $\gamma + \frac{\delta_1}{2} \in A_\epsilon$. This contradicts the fact that γ is an upper bound for A_ϵ, completing Step 3.

Conclusion. By Step 1, $a \le \gamma \le b$. By Step 3, $\gamma \not< b$. So $\gamma = b$. By Step 2, $\gamma \in A_\epsilon$. So $b \in A_\epsilon$. I.E., there is a $\delta > 0$ such that, for all $y, z \in [a, b]$, if $|y - z| < \delta$, then $|f(y) - f(z)| < \epsilon$.

Because $\epsilon > 0$ was arbitrary, this shows that f is uniformly continuous on $[a, b]$. □