THE RIEMANN INTEGRAL

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Reminders on the Darboux integral.

For a bounded function $f\colon [a,b] \to \mathbb{R}$, its lower and upper Darboux sums for a partition $P = \{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \ldots < t_n = b$ are

$$L(f,P) = \sum_{j=1}^{n} m_j (t_j - t_{j-1}) \quad \text{where} \quad m_j = \inf_{[t_{j-1},t_j]} f \qquad \text{and}$$

$$U(f,P) = \sum_{j=1}^{n} M_j (t_j - t_{j-1}) \quad \text{where} \quad M_j = \sup_{[t_{j-1},t_j]} f.$$

The lower and upper Darboux integrals are

$$\underline{\int_a^b} f := \sup_P L(P,f) \qquad \text{and} \qquad \overline{\int_a^b} f := \inf_P U(P,f).$$

By definition, f is Darboux integrable if its lower and upper Darboux integrals are equal, and, if so, we define its integral to be their common value:

$$\int_a^b f := \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Criterion: f is Darboux integrable iff for each $\epsilon > 0$ there exists a partition P such that $|U(f,P) - L(f,P)| < \epsilon$.

The Riemann integral.

A **tagged partition** $(P, \{x_j\})$ of a closed interval $[a,b]$ is a partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ together with a choice of a point $x_j \in [t_{j-1}, t_j]$ for each j. The corresponding **Riemann sum** is

$$S(f, P, \{x_j\}) := \sum_{j=1}^{n} f(x_j)(t_j - t_{j-1}).$$

The Riemann sum satisfies $L(f,P) \leq S(f,P,\{x_j\}) \leq U(f,P)$.

The **mesh** of a partition $P = \{t_0, t_1, \ldots, t_n\}$ where $a = t_0 < t_1 < \ldots < t_n = b$ is

$$|P| := \max_{1 \leq j \leq n} |t_j - t_{j-1}|.$$

Definition: f is **Riemann integrable** with Riemann integral equal to I if it satisfies the following condition:

> For every $\epsilon > 0$ there exists $\delta > 0$ such that, for every tagged partition $(P, \{x_j\})$ with mesh $|P| < \delta$, its Riemann sum satisfies $|S(f, P, \{x_j\}) - I| < \epsilon$.

Theorem: f is Riemann integrable if and only if f is bounded and is Darboux integrable. If this is so, the Riemann integral of f and the Darboux integral of f are equal.

The Appendix to Chapter 13 of Spivak's book contains a proof that if f is bounded and Darboux integrable then f is Riemann integrable with Riemann integral equal to its Darboux integral. The other direction we leave to you as the following two-step exercise.

(1) Let $f\colon [a, b] \to \mathbb{R}$ be any function. Suppose that f is Riemann integrable. Prove that f is bounded.

(Hint: It is enough to show that if f is not bounded from above or from below then it is not Riemann integrable. Suppose that f is not bounded from above; the other case is similar. It is enough to show that, for each $M \in \mathbb{R}$ and for each $\delta > 0$, there is a tagged partition with mesh $< \delta$ for which the Riemann sum of f is $> M$.

Fix $M \in \mathbb{R}$ and $\delta > 0$. Choose an arbitrary partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with mesh $< \delta$. Show that there is at least one interval $[t_{j-1}, t_j]$ of the partition on which f is not bounded from above. Choose arbitrary points $x_{j'} \in [t_{j'-1}, t_{j'}]$ for all $j' \neq j$. Finally, choose $x_j \in [t_{j-1}, t_j]$ such that $f(x_j)$ is sufficiently large.)

(2) Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable. Prove that f is Darboux integrable.

(Hint: The Riemann integral of f is $\leq$ the upper Darboux integral of f and $\geq$ the lower Darboux integral of f. We claim that the opposite equalities hold too. Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$, and let m_j and M_j be the infimum and supremum of the values of f on $[t_{j-1}, t_j]$. For any $\epsilon > 0$, there exists $x_j \in [t_{j-1}, t_j]$ such that $f(x_j) > M_j - \epsilon$; the corresponding Riemann sum satisfies $S(f, P, \{x_j\}) > U(f, P) - \epsilon(b - a)$. It follows that the Riemann integral of f is $\geq$ the upper Darboux integral of f. By a similar argument, the Riemann integral of f is also $\leq$ the lower Darboux integral of f.)