[1]

LENGTH OF A CURVE

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Drawing diagrams while reading this note will much help you to follow the note.

A (parametrized) curve in the plane is a continuous map $\gamma\colon I \to \mathbb{R}^2$ from an interval I to the plane $\mathbb{R}^2$. See Spivak's appendix to Chapter 12. We can write the curve as $\gamma(t) = (x(t), y(t))$ for $x, y\colon I \to \mathbb{R}$. *Continuous* has an ϵ-δ definition using the notion of *distance* in the plane and is equivalent to x and y being continuous as functions from I to $\mathbb{R}$.

A *partition* of an interval $I = [a, b]$ is a set of points that can be written as $P = \{t_0, t_1, \ldots, t_n\}$ where $a = t_0 < t_1 < \ldots < t_n = b$.

Fix a curve in the plane,

$$\gamma\colon [a, b] \to \mathbb{R}^2 \quad , \quad \gamma(t) = (x(t), y(t)).$$

For any partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$, consider the polygonal line through the points $\gamma(t_0)$, $\gamma(t_1)$, $\ldots$, $\gamma(t_n)$, and define

$$\ell(\gamma, P) := \sum_{j=1}^{n} \|\gamma(t_{j-1}) - \gamma(t_j)\|.$$

Definition. γ is **rectifiable** if the set of numbers

$$\{\ell(\gamma, P) \mid P \text{ is a partition of } [a, b]\}$$

is bounded. If so, define the length of the curve γ to be the supremum of this set:

$$\text{length}(\gamma) := \sup_P \ell(\gamma, P).$$

Exercise (Characterization of length)**.**

(i) The following two properties of γ are equivalent.

  (i a) γ is rectifiable, and $\text{length}(\gamma) \leq c$.

  (i b) For every partition P of I, we have $\ell(\gamma, P) \leq c$.

(ii) The following two properties of γ are equivalent.

  (ii a) γ is not rectifiable, or $\text{length}(\gamma) \geq c$.

  (ii b) For every $\epsilon > 0$, there exists a partition P such that $\ell(\gamma, P) > c - \epsilon$.

Definition. Given two partitions P and Q of the same closed interval I, Q is a **refinement** of P if every point of P is also in Q.

[1] As always, please let me know if you find typos.

Lemma. *If Q is a refinement of P, then $\ell(\gamma, P) \leq \ell(\gamma, Q)$.*

Proof. Let Q be a refinement of P. Then Q can be obtained from P by adding finitely many points.

Consider first the special case in which Q is obtained from P by adding one point: $P = \{t_0, \dots, t_n\}$, and $Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\}$ where $a = t_0 < t_1 < \dots < t_{k-1} < u < t_k < \dots < t_n = b$. By the triangle inequality,

$$\|\gamma(t_{k-1}) - \gamma(t_k)\| \leq \|\gamma(t_{k-1}) - \gamma(u)\| + \|\gamma(u) - \gamma(t_k)\|.$$

Adding $\sum_{j=1}^{k-1} \|\gamma(t_{j-1}) - \gamma(t_j)\|$ and $\sum_{j=k+1}^{n} \|\gamma(t_{j-1}) - \gamma(t_j)\|$ to both sides, we obtain the inequality $\ell(\gamma, P) \leq \ell(\gamma, Q)$.

The case that Q is obtained from P by adding m points follows by induction on m. □

We define $\boldsymbol{\pi}$ to be the length of a half-circle. The following example shows that π is between 3 and 4.

Example. Consider the portion of the unit circle $\{x^2 + y^2 = 1\}$ that lies to the right of the y axis. Parametrize it as

$$\gamma(t) = (\sqrt{1 - t^2}, t) \quad , \quad -1 \leq t \leq 1.$$

Then γ is rectifiable, and $3 \leq \text{length}(\gamma) \leq 4$.

Proof. It is enough to find a partition P such that $\ell(\gamma, P) \geq 3$, and to show that for every partition P we have $\ell(\gamma, P) \leq 4$. (Please make sure that you understand why this is enough.)

Consider the partition P of $[-1, 1]$ whose points are $-1 < -\frac{1}{2} < \frac{1}{2} < 1$. Then $\ell(\gamma, P) = 3$. (Please check this!)

Now let P be an arbitrary partition of $[-1, 1]$. We would like to show that $\ell(\gamma, P) \leq 4$.

We begin with the special case that the partition P contains the point 0. In this case, we can write the points of P as $-1 = t_0 < \dots < t_k < \dots < t_n = 1$ with $t_k = 0$. Then

$$\ell(\gamma, P) = \ell(\gamma_1, P_1) + \ell(\gamma_2, P_2),$$

where γ_1 and γ_2 are the restrictions of γ to the subintervals $[-1, 0]$ and $[0, 1]$, where P_1 is the partition of $[-1, 0]$ whose points are $-1 = t_0 < t_1 < \dots < t_k = 0$, and where P_2 is the partition of $[0, 1]$ whose points are $0 = t_k < t_{k+1} < \dots < t_n = 1$. (Exercise: check this.)

Write $\gamma(t_j) = (x_j, y_j)$ and $R_j = (x_j, y_{j-1})$. Then

$$(\star) \qquad \begin{aligned} \|\gamma(t_{j-1}) - \gamma(t_j)\| &\leq \|\gamma(t_{j-1}) - R_j\| + \|R_j - \gamma(t_j)\| \quad \text{by the triangle inequality} \\ &= |x_j - x_{j-1}| + |y_j - y_{j-1}| \quad \text{(exercise)} . \end{aligned}$$

The sequence $y_j = t_j$ is increasing for $j = 0, 1, \dots, n$. The sequence $x_j = \sqrt{1 - t_j^2}$ is increasing for $j = 0, 1, \dots, k$ and decreasing for $j = k, k+1, \dots, n$. So the right hand side of $(\star)$ is equal to $(x_j - x_{j-1}) + (y_j - y_{j-1})$ if $1 \leq j \leq k$ and to $(x_{j-1} - x_j) + (y_j - y_{j-1})$ if

$k+1 \le j \le n$. Summing the left and right hand sides of ($\star$) over $j \in \{1, \ldots, k\}$, the left hand side adds up to $\ell(\gamma_1, P_1)$, and the right hand side becomes a telescopic sum:

$$\ell(\gamma_1, P_1) \le \sum_{j=1}^{k} (x_j - x_{j-1}) + \sum_{j=1}^{k} (y_j - y_{j-1})$$
$$= (x_k - x_0) + (y_k - y_0) = (1-0) + (0-(-1)) = 2.$$

Similarly, summing the left and right hand sides of ($\star$) over $j \in \{k+1, \ldots, n\}$ yields

$$\ell(\gamma_2, P_2) \le \sum_{j=k+1}^{n} (x_{j-1} - x_j) + \sum_{j=k+1}^{n} (y_j - y_{j-1})$$
$$= (x_k - x_n) + (y_n - y_k) = (1-0) + (1-0) = 2.$$

Thus, $\ell(\gamma, P) = \underbrace{\ell(\gamma_1, P_1)}_{\le 2} + \underbrace{\ell(\gamma_2, P_2)}_{\le 2} \le 4$.

If P does not contain the point 0, let P' be obtained from P by adding the point 0. Applying the special case that we proved to the partition P', we obtain $\ell(\gamma, P') \le 4$. By the lemma, $\ell(\gamma, P) \le \ell(\gamma, P')$. From these two inequalities we obtain $\ell(\gamma, P) \le 4$. □

Please make sure that you understand the properties of "length" that are described in the following exercises.

Exercise (concatenation). Let $a, b, c \in \mathbb{R}$ be such that $a < b < c$. Let $\gamma \colon [a, c] \to \mathbb{R}^2$ be a curve. Let $\gamma_1 \colon [a, b] \to \mathbb{R}^2$ and $\gamma_2 \colon [b, c] \to \mathbb{R}^2$ be the restrictions of γ to the subintervals $[a, b]$ and $[b, c]$. Show that γ is rectifiable if and only if γ_1 and γ_2 are rectifiable and, if so, then $\text{length}(\gamma) = \text{length}(\gamma_1) + \text{length}(\gamma_2)$.

Hint:

(1) If P_1 is a partition of $[a, b]$ and P_2 is a partition of $[b, c]$, then $P_1 \cup P_2$ is a partition of $[a, b]$, and $\ell(\gamma, P_1 \cup P_2) = \ell(\gamma_1, P_1) + \ell(\gamma_2, P_2)$. Conclude that if γ is rectifiable then so are γ_1 and γ_2,oand $\text{length}\,\gamma \ge \text{length}\,\gamma_1 + \text{length}\,\gamma_2$.

(2) If P is a partition of $[a, c]$ and $Q := P \cup \{b\}$, then $P_1 := Q \cap [a, b]$ is a partition of $[a, b]$ and $P_2 := Q \cap [b, c]$ is a partition of $[b, c]$, and $\ell(\gamma, P) \le \ell(\gamma_1, P_1) + \ell(\gamma_2, P_2)$. Conclude that if γ_1 and γ_2 are rectifiable then so is γ, and $\text{length}(\gamma) \le \text{length}(\gamma_1) + \text{length}(\gamma_2)$.

Exercise (Invariance under reparametrization). Let $a, b, c, d \in \mathbb{R}$ be such that $a < b$ and $c < d$. Let

$$\gamma \colon [a, b] \to \mathbb{R}^2$$

be a curve. Let

$$h \colon [c, d] \to [a, b]$$

be a continuous function that is strictly monotone increasing and takes c to a and d to b, and consider the curve

$$\gamma \circ h \colon [c, d] \to \mathbb{R}^2.$$

Show that γ is rectifiable if and only if $\gamma \circ h$ is rectifiable, and, if so, then $\text{length}(\gamma) = \text{length}(\gamma \circ h)$.

Hint:

(1) By a theorem that we proved about strictly increasing continuous functions, the image of h is $[a, b]$, and the inverse function $h^{-1}\colon [a, b] \to [c, d]$ is also a strictly increasing continuous function.

(2) If $Q = \{\tau_0, \tau_1, \ldots, \tau_n\}$ is a partition of $[c, d]$, and if we define $t_j := h(\tau_j)$ for each j, then $P := \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$, and $\ell(\gamma, P) = \ell(\gamma \circ h, Q)$. Conclude that $\text{length}(\gamma) \geq \text{length}(\gamma \circ h)$.

(3) If $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$, and if we define $\tau_j := h^{-1}(t_j)$ for each j, then $Q := \{\tau_0, \tau_1, \ldots, \tau_n\}$ is a partition of $[c, d]$, and $\ell(\gamma \circ h, Q) = \ell(\gamma, P)$. Conclude that $\text{length}(\gamma \circ h) \geq \text{length}(\gamma)$.

Exercise (Invariance under rotations). Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map $R(x, y) := (\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}})$. This map takes the origin to itself, and it preserves distances in the sense that, for any two points p and q, we have $\|R(p) - R(q)\| = \|p - q\|$. (Geometrically, this map is a rotation by 45 degrees about the origin. Can you see why?) Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a curve. Consider the curve $R \circ \gamma\colon [a, b] \to \mathbb{R}^2$. Then γ is rectifiable if and only if $R \circ \gamma$ is rectifiable, and, if so, $\text{length}(\gamma) = \text{length}(R \circ \gamma)$.

Hint: For any partition $P = \{t_0, \ldots, t_n\}$ of $[a, b]$, we have $\ell(\gamma, P) = \ell(R \circ \gamma, P)$.

Lemma. *Suppose that (γ is continuous and) $\gamma\colon [a, b] \to \mathbb{R}^2$ is rectifiable. Then*

$$L(t) := \text{length}(\gamma|_{[a,t]})$$

(and $L(a) := 0$) is continuous from the left at b and from the right at a.

Proof. By concatenation,

$$L\colon [a, b] \to [0, \infty)$$

is non-decreasing. Let $\epsilon > 0$.

Let $\epsilon_1 = \frac{1}{2}\epsilon$. Let $\delta > 0$ be such that for all $t \in (b - \delta, b]$

$$\|\gamma(t) - \gamma(b)\| < \epsilon_1.$$

Let $c = \text{length}(\gamma)$.

Let $\epsilon_2 = \frac{1}{2}\epsilon$. Let $P = \{t_0, \ldots, t_n\}$ with $a = t_0 < \ldots < t_{n-1} < t_n = b$ be a partition of $[a, b]$ such that $\ell(\gamma, P) > c - \epsilon_2$.

Then for all τ such that

$$\max\{t_{n-1}, b - \delta\} < \tau \leq b,$$

we have

$$
\begin{aligned}
L(\tau) &= \text{length}(\gamma|_{[a,\tau]}) \\
&\geq \left(\sum_{j=1}^{n-1} \|\gamma(t_{j-1}) - \gamma(t_j)\|\right) + \|\gamma(t_{n-1}) - \gamma(\tau)\| \\
&\qquad\qquad\qquad \text{by considering the partition } a = t_0 < \ldots < t_{n-1} < \tau \text{ of } [a,\tau] \\
&= \left(\sum_{j=1}^{n-1} \|\gamma(t_{j-1}) - \gamma(t_j)\|\right) \\
&\qquad + \underbrace{\|\gamma(t_{n-1}) - \gamma(\tau)\| + \|\gamma(\tau) - \gamma(b)\|}_{\geq \|\gamma(t_{n-1}) - \gamma(b)\| \text{ by the triangle inequality}} - \underbrace{\|\gamma(\tau) - \gamma(b)\|}_{< \epsilon_1 \text{ since } \tau \in (b-\delta, b]} \\
&> \underbrace{\left(\sum_{j=1}^{n} \|\gamma(t_{j-1}) - \gamma(t_j)\|\right)}_{\substack{= \ell(\gamma, P) \\ > c - \epsilon_2 \text{ by the choice of } P}} - \epsilon_1 \\
&> c - \epsilon_1 - \epsilon_2 \\
&= c - \epsilon \quad \text{ by the choice of } \epsilon_1 \text{ and } \epsilon_2.
\end{aligned}
$$

Thus, L is left continuous at b. A similar argument shows that L is right continuous at a. □

Theorem. *Suppose that* $\gamma\colon [a,b] \to \mathbb{R}^2$ *is (continuous and) rectifiable. Then* $L(t) := \text{length}(\gamma|_{[a,t]})$ *is continuous.*

Proof. It is enough to prove that L is continuous from the left at x for all $x \in (a,b]$ and continuous from the right at x for all $x \in [a,b)$.

If $x \in (a,b]$ then by the lemma applied to $\gamma|_{[a,x]}$ $L(t)$ is continuous from the left at x.

If $x \in [a,b)$ then for all $t \in [x,b]$ we have

$$L(t) = \underbrace{\text{length}(\gamma|_{[a,x]})}_{=\text{constant}} + \text{length}(\gamma|_{[x,t]})$$

by concatenation. By the lemma applied to $\gamma|_{[x,b]}$ the 2nd summand, hence $L(t)$, is continuous from the right at x.

□