LENGTH OF A CONTINUOUSLY DIFFERENTIABLE CURVE

MAT157, WINTER 2021. YAEL KARSHON

We say that a function is **continuously differential** if it is differentiable and its derivative is continuous.

Theorem. *Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a curve. Write $\gamma(t) = (x(t), y(t))$. Assume that the functions $x, y\colon [a,b] \to \mathbb{R}$ are C^1; denote their derivatives by $\dot{x}$ and $\dot{y}$. Define $s\colon [a,b] \to \mathbb{R}$ by $s := \sqrt{\dot{x}^2 + \dot{y}^2}$; call it the **speed** of γ. Then* $\text{length}(\gamma) = \int_a^b s(t)dt$.

We will now give two lemmas. We will use Lemma 1 to prove Lemma 2, and we will use Lemma 2 to prove the theorem.

Lemma 1. *Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every ρ, σ, τ that are contained in a subinterval of $[a,b]$ of length $< \delta$, we have $\left|\sqrt{f(\rho)^2 + g(\sigma)^2} - \sqrt{f(\tau)^2 + g(\tau)^2}\right| < \epsilon$.*

Proof of Lemma 1. Let $\epsilon > 0$.

Let $M := \max\limits_{[a,b]} f^2 + \max\limits_{[a,b]} g^2$. The square root function on the closed interval $[0, M]$ is continuous, hence uniformly continuous. Let $\epsilon_1 > 0$ be such that, for every $r_1, r_2 \in [0, M]$, if $|r_1 - r_2| < \epsilon_1$, then $\left|\sqrt{r_1} - \sqrt{r_2}\right| < \epsilon$.

The functions f^2 and g^2 on the closed interval $[a,b]$ are continuous, hence uniformly continuous. Let $\delta > 0$ be such that, for every $\tau, \hat{\tau} \in [a,b]$, if $|\hat{\tau} - \tau| < \delta$, then $|f(\hat{\tau})^2 - f(\tau)^2| < \dfrac{\epsilon_1}{2}$ and $|g(\hat{\tau})^2 - g(\tau)^2| < \dfrac{\epsilon_1}{2}$.

Let ρ, σ, τ be in a subinterval of $[a,b]$ of length $< \delta$. Let $r_1 := f(\rho)^2 + g(\sigma)^2$ and $r_2 := f(\tau)^2 + g(\tau)^2$. Then $r_1, r_2 \in [0, M]$, and we have $|r_1 - r_2| = |f(\rho)^2 - f(\tau)^2 + g(\sigma)^2 - g(\tau)^2| \leq |f(\rho)^2 - f(\tau)^2| + |g(\sigma)^2 - g(\tau)^2|$, which, by the choice of δ, is $< \epsilon_1$. By the choice of ϵ_1, it follows that $\left|\sqrt{r_1} - \sqrt{r_2}\right| < \epsilon$, which proves the lemma. □

Lemma 2. *Let $\gamma\colon [a,b] \to \mathbb{R}^2$ and $s\colon [a,b] \to \mathbb{R}$ be as in the statement of the theorem. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every partition P with mesh $< \delta$, we have $\left|\ell(\gamma, P) - \int_a^b s(t)dt\right| < \epsilon$.*

Proof of Lemma 2. Let $\epsilon > 0$ be arbitrary. Let $\widehat{\epsilon} := \dfrac{\epsilon}{b - a + 1}$.

Because the (Darboux) integral $\int_a^b s(t)dt$ is equal to the Riemann integral, and by the definition of the Riemann integral, we can find $\delta_1 > 0$ such that, for every tagged partition

[0] As usual, please let me know if you find typos.

$(P, \{\tau_j\})$ with mesh $< \delta_1$, the corresponding Riemann sum $S(s, P, \{\tau_j\})$ satisfies

$$\left| \int_a^b s(t)dt - S(s, P, \{\tau_j\}) \right| < \widehat{\epsilon}.$$

By Lemma 1, we can find $\delta_2 > 0$ such that, for every ρ, σ, τ that are contained in a subinterval of $[a, b]$ of length $< \delta_2$, we have $\left| \sqrt{\dot{x}(\rho)^2 + \dot{y}(\sigma)^2} - \sqrt{\dot{x}(\tau)^2 + \dot{y}(\tau)^2} \right| < \widehat{\epsilon}$, that is, $\left| \sqrt{\dot{x}(\rho)^2 + \dot{y}(\sigma)^2} - s(\tau) \right| < \widehat{\epsilon}$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Let $(P, \{\tau_j\})$ be any tagged partition of $[a, b]$ with mesh $< \delta$. Write $P = \{t_0, t_1, \ldots, t_n\}$, and write $\gamma(t_j) = (x_j, y_j)$. By the mean value theorem for the functions $x(t)$ and $y(t)$ on the intervals $[t_{j-1}, t_j]$, for every j there exist $\rho_j, \sigma_j \in [t_{j-1}, t_j]$ such that

$$x_j - x_{j-1} = (t_j - t_{j-1})\dot{x}(\rho_j) \quad \text{and} \quad y_j - y_{j-1} = (t_j - t_{j-1})\dot{y}(\sigma_j).$$

Squaring, summing, and taking the square root, we get

$$\sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2} = (t_j - t_{j-1})\sqrt{\dot{x}(\rho_j)^2 + \dot{y}(\sigma_j)^2}.$$

For every j, since the points ρ_j, σ_j, τ_j are contained in the subinterval $[t_{j-1}, t_j]$ that has length $< \delta_2$, we have $\left| \sqrt{\dot{x}(\rho_j)^2 + \dot{y}(\sigma_j)^2} - s(\tau_j) \right| < \widehat{\epsilon}$. By the above equation,

$$(t_j - t_{j-1})(s(\tau_j) - \widehat{\epsilon}) \ < \ \sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2} \ < \ (t_j - t_{j-1})(s(\tau_j) + \widehat{\epsilon}).$$

Summing over j, and by the definitions of $\ell(\gamma, P)$ and $S(s, P, \{\tau_j\})$, we get

$$S(s, P, \{\tau_j\}) - \widehat{\epsilon}(b - a) \ < \ \ell(\gamma, P) \ < \ S(s, P, \{\tau_j\}) + \widehat{\epsilon}(b - a).$$

Since the Riemann sum $S(s, P, \{\tau_j\})$ is $\widehat{\epsilon}$-close to $\int_a^b s(t)dt$, and since $(b - a + 1)\widehat{\epsilon} = \epsilon$ by the choice of $\widehat{\epsilon}$, the above equation implies that $\ell(\gamma, P)$ is ϵ-close to $\int_a^b s(t)dt$, as required. $\square$

We are now ready to prove the theorem. Below, when we write "length$(\gamma) \leq \ell$", we mean that γ is rectifiable and length$(\gamma) \leq \ell$. Similarly, when we write "length$(\gamma) \geq \ell$", we mean that either γ is rectifiable and length$(\gamma) \geq \ell$, or that γ is not rectifiable.

Proof of the theorem. Let $\epsilon > 0$. Let $\delta > 0$ be as in Lemma 2. Let P be a partition of $[a, b]$ of mesh $< \delta$. Then

$$\text{length}(\gamma) \geq \ell(\gamma, P) > \int_a^b s(t)dt - \epsilon,$$

where the first inequality is by the definition of length(γ) and the second inequality is by the choice of δ. Since $\epsilon > 0$ was arbitrary, length$(\gamma) \geq \int_a^b s(t)dt$.

Let P be any partition. Let Q be a refinement of P with mesh $< \delta$. Then

$$\ell(\gamma, P) \leq \ell(\gamma, Q) < \int_a^b s(t)dt + \epsilon,$$

where the first inequality is because Q is a refinement of P and the second is by the choice of δ. Since P was arbitrary, length$(\gamma) \leq \int_a^b s(t)dt + \epsilon$. Since $\epsilon > 0$ was arbitrary, length$(\gamma) \leq \int_a^b s(t)dt$. $\square$