THE LOGARITHM AND EXPONENTIAL FUNCTIONS

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We would like to define log and exp to be inverses of each other, such that the derivative of exp is exp, and such that $\exp(0) = 1$. By the formula for the derivative of an inverse function, the derivative of $\log(x)$ should then be $\frac{1}{x}$. (Make sure that you see why!).

Definition of log. For $x > 0$,

$$\log x = \int_1^x \frac{1}{t} dt.$$

Remark. If $0 < x < 1$ then, by definition, $\displaystyle\int_1^x \frac{1}{t} dt = -\int_x^1 \frac{1}{t} dt.$

Properties of log.

(i) $\log x$ is positive if $x > 1$, negative if $0 < x < 1$, and zero if $x = 1$.

(ii) log is continuous and strictly monotone increasing.

(iii) $\displaystyle\frac{d}{dx} \log x = \frac{1}{x}$.

(iv) $\log(xy) = \log(x) + \log(y)$. Consequently, $\log(1/x) = -\log x$, and for any $n \in \mathbb{N}$ we have $\log(x^n) = n \log x$.

(v) $\displaystyle\lim_{x\to\infty} \log(x) = \infty$; $\displaystyle\lim_{\substack{x\to 0\\ x>0}} \log(x) = -\infty$.

Proof. (i) and (ii) follow from the properties of definite integrals, and (iii) follows from the fundamental theorem of calculus. (Please check the details, also in the case that $0 < x < 1$.)

Fix $y > 0$, and consider the function $f(x) := \log(xy) - \log x$, for $x > 0$. We have $\frac{d}{dx} f(x) = \frac{d}{dx} (\log(xy) - \log x) = \frac{1}{xy} y - \frac{1}{x} = 0$, where the second equality is by the chain rule. So $f(x)$ is constant. Since $f(1) = \log y$ and f is constant, $f(x) = \log y$ for all x. That is, $\log(xy) = \log x + \log y$ for all x. Since y was arbitrary, this proves (iv).

The conclusion $\log(1/x) = -\log x$ is obtained by setting $y = 1/x$. The conclusion $\log(x^n) = n \log x$ is proved by induction on n, using the recursive definition of x^n for $n \in \mathbb{N}$.

To prove (v), let $M \in \mathbb{R}$. Let n be an integer that is greater than $M/\log 2$; such an integer exists by the Archimedean property of the real numbers. For all $x > 2^n$ we have $\log x > \log(2^n) = n \log 2 > M$. Because M was arbitrary, this proves that $\displaystyle\lim_{x\to\infty} \log x = \infty$.

Similarly, given any $M \in \mathbb{R}$, if n is an integer that is greater than $-M/\log 2$, then for all $0 < x < 1/2^n$ we have $\log x < \log(1/2^n) = -n \log 2 < M$; because M was arbitrary, this proves that $\displaystyle\lim_{x\to 0} \log x = -\infty$. $\square$

Properties (ii) and (v) imply that the function $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$ is one-to-one and onto, so it has an inverse function that is defined on all of $\mathbb{R}$ and whose image is $\mathbb{R}_{>0}$.

Definition of exp. The function exp is the inverse function of log.

Properties of exp.

(i) $\exp(x) > 1$ if x is positive, $0 < \exp(x) < 1$ if x is negative, and $\exp(0) = 1$.

(ii) exp is monotone increasing.

(iii) $\frac{d}{dx} \exp x = \exp x$.

(iv) $\exp(x + y) = \exp(x) \exp(y)$.

Proof. (i), (ii), and (iii) follow from the corresponding properties of log (please check the details). For (iv), take any x and y in $\mathbb{R}$. Because log is onto, there exist α and β such that $x = \log \alpha$ and $y = \log \beta$. (iv) then follows from $\log(\alpha\beta) = \log \alpha + \log \beta$ (please check the details). $\square$

Growth of exp.

(i) For all $x \in \mathbb{R}$, $\exp x \geq 1 + x$.

(ii) For all $x > 0$ and all $n \in \mathbb{N}$, $\exp x \geq 1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!}$.

(iii) For all $n \in \mathbb{N}$, $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$.

(iv) For all $n \in \mathbb{N}$, $\displaystyle\lim_{x\to 0+} \frac{e^{-1/x}}{x^n} = 0$.

(v) The function

$$f(x) = \begin{cases} e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is infinitely differentiable and its derivatives to all orders vanish at $x = 0$.

Proof. For (i) with $x > 0$, write $\exp x - 1 = \int_0^x \exp t dt$; the integral is $\geq x$ because $\exp t \geq 1$. For (i) with $x < 0$, write $\exp x - 1 = -\int_x^0 \exp t dt$; here $\exp t \leq 1$, so the integral is $\leq |x|$, and its negative is $\geq x$ as required.

(ii) is proved by induction on n. The case $n = 1$ is in (i). As before, write $\exp x - 1 = \int_0^x \exp(t) dt$. Assuming the inequality for n and substituting it in the integrand, we get $\exp x - 1 \geq \int_0^x (1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}) dt$; evaluating the integral, we get the inequality for $n+1$.

(ii) implies that for all $x > 0$ and $n \in \mathbb{N}$ we have the inequality $e^x > \frac{x^{n+1}}{(n+1)!}$. To obtain (iii) from this inequality, we divide both sides by x^n and note that the resulting right hand side diverges to ∞ as $x \to \infty$. To obtain (iv) from the inequality, we substitute in it $1/x$ instead of x; after a minor manipulation this becomes the inequality $\frac{e^{-1/x}}{x^n} \leq (n+1)!x$, whose right hand side converges to 0 when $x \to 0+$. (Please check the details.)

We now show (v). For every $k \in \mathbb{Z}_{\geq 0}$ there exists a rational function $r_k(x)$ (i.e., a ratio of two polynomials) such that $f^{(k)}(x) = r_k(x)e^{-1/x}$ for $x > 0$. Indeed, for $k = 0$ we can take $r_0(x) = 1$, and for higher k this follows by induction on k (please check the details). For any rational function $r(x)$ we have $\lim_{x\to 0+} r(x)e^{-1/x} = 0$; this is a consequence of (iv) (please check the details). From these two facts we obtain that for all $k \in \mathbb{Z}_{\geq 0}$ we have $\lim_{x\to 0+} f^{(k)}(x)/x = 0$. A similar argument shows that $\lim_{x\to 0-} f^{(k)}(x)/x = 0$ (please check this; note that the absolute value in the definition of f is important). We get that $\lim_{x\to 0} f^{(k)}(x)/x = 0$ (because the left and right limits are both zero). We can now use this to prove by induction that all the derivatives $f^{(k)}(0)$ exist and are equal to zero. The base case, $f(0) = 0$, follows from the definition of f. For the induction step, assume that $f^{(k)}(0) = 0$; then the difference quotient $\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0}$ is equal to $f^{(k)}(x)/x$, which converges to 0 as $x \to 0$, and so $f^{(k+1)}(0)$ exists and is equal to 0. $\square$

Remark. The function that is given by e^{-1/x^2} when $x \neq 0$ and 0 when $x = 0$ has similar properties: it is infinitely differentiable and all its derivatives vanish at $x = 0$.

Exponentiation. For $a > 0$ and $b \in \mathbb{R}$, we define

$$a^b := \exp(b \log(a)).$$

This is consistent with the recursive definition of a^n for $n \in \mathbb{N}$, because with our new definition we still have $a^0 = 1$ and $a^{b+1} = a^b \cdot a$.

In fact, we have all the usual properties of exponentials: $a^0 = 1$, $a^1 = a$, $a^{b+c} = a^b \cdot a^c$, $(ab)^c = a^c \cdot b^c$, $\left(a^b\right)^c = a^{bc}$, $a^{-b} = 1/a^b$, and $a^{1/m} = \sqrt[m]{a}$ for $m \in \mathbb{N}$. Also, $\frac{d}{dx}x^b = bx^{b-1}$ and $\frac{d}{dx}a^x = a^x \log a$. All of these follow from this new definition of exponentiation by the properties of exp and log.

Finally, we define

$$e := \exp(1).$$

Then for all x, we have $e^x = \exp(x)$.