LIM-INF AND LIM-SUP

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Definition. A sequence $(a_n)_{n=1}^{\infty}$ converges to 7 if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n > N$ we have $|a_n - 7| < \epsilon$.

Theorem (Monotone convergence theorem).

- *If a sequence (a_n) is weakly increasing and is bounded from above, then the sequence converges to its supremum.*
- *If a sequence (a_n) is weakly decreasing and is bounded from below, then the sequence converges to its infimum.*

Theorem (Sandwich theorem). *Let (a_n), (b_n), and (c_n) be sequences. Assume that (a_n) and (c_n) both converge to 7 and that $a_n \leq b_n \leq c_n$ for all n. Then (b_n) also converges to 7.*

Exercise. Prove the monotone convergence theorem and the sandwich theorem.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $\{a_n\}_{n=1}^{\infty}$ is bounded from below by -11 and from above by 19. Then the "tail" $\{a_k\}_{k\geq 15}$ is also bounded, from below by -11 and from above by 19, so it has an infimum m_{15} and a supremum M_{15}, which are between -11 and 19. Moreover, for every subset of $\{a_k\}_{k\geq 15}$, in particular for the subset $\{a_k\}_{k\geq 20}$, its supremum is $\leq M_{15}$ and its infimum is $\geq m_{15}$.

More generally, given any bounded sequence of real numbers $(a_n)_{n=1}^{\infty}$, for every $n \in \mathbb{N}$ let

$$m_n := \inf\{a_k\}_{k\geq n} \quad \text{and} \quad M_n := \sup\{a_k\}_{k\geq n}.$$

Then (m_n) is weakly increasing, (M_n) is weakly decreasing, and any lower and upper bounds of the sequence (a_n) are also lower and upper bounds for the sequences (m_n) and (M_n). By the monotone convergence theorem, the sequences (m_n) and (M_n) have limits.

Definition. The lim-sup and the lim-inf of (a_n) are the limits

$$\overline{\lim_{n\to\infty}}\, a_n := \lim_{n\to\infty} M_n \quad \text{and} \quad \underline{\lim_{n\to\infty}}\, a_n := \lim_{n\to\infty} m_n.$$

Example. Let $a_n = (-1)^n - \dfrac{1}{n}$.

n	1	2	3	4	5	6	7	...
a_n	-2	$\frac{1}{2}$	$-\frac{4}{3}$	$\frac{3}{4}$	$-\frac{6}{5}$	$\frac{5}{6}$	$-\frac{8}{7}$	...
M_n	1	1	1	1	1	1	1	...
m_n	-2	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{6}{5}$	$-\frac{6}{5}$	$-\frac{8}{7}$	$-\frac{8}{7}$	...

Lemma. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of real numbers. Suppose that for all* $n > 200$ *we have* $-5 \le a_n \le 77$. *Then* $\{a_n\}$ *is bounded, and* $-5 \le \underline{\lim}\, a_n \le \overline{\lim}\, a_n \le 77$.

Proof. The number $\max\{a_1, \ldots, a_{200}, 77\}$ is an upper bound for the sequence, and the number $\min\{a_1, \ldots, a_{200}, -5\}$ is a lower bound for the sequence. So the sequence is bounded.

Let $n > 200$. Then the "tail" $\{a_k\}_{k \ge n}$ is bounded from below by -5 and from above by 77. So its infimum and supremum satisfy $-5 \le m_n \le M_n \le 77$. Taking the limits as $n \to \infty$, these equalities imply that $-5 \le \underline{\lim}\, a_n \le \overline{\lim}\, a_n \le 77$. □

Theorem. *Let* $(a_n)_{n \in \mathbb{N}}$ *be a sequence of real numbers. Then* (a_n) *converges if and only if it is bounded and* $\underline{\lim}\, a_n = \overline{\lim}\, a_n$.

Proof of "if": Assume that $\{a_n\}$ is bounded and that $\underline{\lim}\, a_n = \overline{\lim}\, a_n$.

Let $\ell := \underline{\lim}\, a_n = \overline{\lim}\, a_n$. Let $M_n = \sup\{a_k\}_{k \ge n}$ and $m_n = \inf\{a_k\}_{k \ge n}$. Then, for every n, we have

$$m_n \le a_n \le M_n.$$

Since (M_n) and (m_n) converge to ℓ, by the "sandwich theorem" (a_n) also converges to ℓ. □

Proof of "only": Assume that (a_n) converges to ℓ.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that for all $n > N$ we have $\ell - \epsilon < a_n < \ell + \epsilon$. By a variant of the lemma (with 200 replaced by N and with -5 and 77 replaced by $\ell - \epsilon$ and $\ell + \epsilon$), we conclude that the sequence (a_n) is bounded and that $\ell - \epsilon \le \underline{\lim}\, a_n \le \overline{\lim}\, a_n \le \ell + \epsilon$. So $|\underline{\lim}\, a_n - \ell| < \epsilon$ and $|\overline{\lim}\, a_n - \ell| < \epsilon$. Because this is true for all $\epsilon > 0$, we conclude that $\underline{\lim}\, a_n = \ell$ and that $\overline{\lim}\, a_n = \ell$. □

Definition. A **Cauchy sequence** is a sequence $(a_n)_{n=1}^{\infty}$ such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every n and m greater than N we have $|a_m - a_n| < \epsilon$.

Theorem (Cauchy criterion for convergence)**.** *For any sequence of real numbers, the sequence converges if and only if it is a Cauchy sequence.*

Proof of "if": Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of real numbers.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that for every m and n greater than N we have $|a_m - a_n| < \epsilon$. Taking $m = N + 1$, for every $n > N$ we have $a_{N+1} - \epsilon < a_n < a_{N+1} + \epsilon$. By a variant of the lemma, we obtain that the sequence is bounded and that $a_{N+1} - \epsilon \le \underline{\lim}\, a_n \le \overline{\lim}\, a_n \le a_{N+1} + \epsilon$. This implies that $|\overline{\lim}\, a_n - \underline{\lim}\, a_n| \le 2\epsilon$. Because $\epsilon > 0$ is arbitrary, this implies that $\underline{\lim}\, a_n = \overline{\lim}\, a_n$. By the previous theorem, the sequence converges. □

Proof of "only if": Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers that converges to a limit ℓ.

Let $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that for all $n > N$ we have $|a_n - \ell| < \frac{1}{2}\epsilon$. Let m and n be greater than N. Then $|a_m - a_n| \le \underbrace{|a_m - \ell|}_{< \frac{1}{2}\epsilon} + \underbrace{|\ell - a_n|}_{< \frac{1}{2}\epsilon} < \epsilon$.

(Here, the first inequality follows from the triangle inequality.) □