CRITERIA FOR CONVERGENCE OF SERIES

MAT157, WINTER 2021. YAEL KARSHON

Does the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ converge or diverge?

Theorem (Integral test)**.** *Let $f\colon [1,\infty) \to \mathbb{R}$ be positive and weakly decreasing. Then either the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f$ both converge, or they both diverge to infinity.*

Since $\int\limits_1^x \frac{1}{t} dt = \log t|_1^x = \log x$ diverges to infinity as $x \to \infty$, by the integral test the harmonic series diverges.

Proof of the integral test.

Recall: "$\int\limits_1^{\infty} f$ converges" means that the function $\underbrace{\int\limits_1^x f}_{=:F(x)}$ has a limit as $x \to \infty$.

Since $f > 0$, the function $F\colon [1,\infty) \to \mathbb{R}$ is increasing. So if F is bounded then F has a limit as $x \to \infty$, and if F is not bounded then F diverges to infinity as $x \to \infty$.

Recall: "$\sum\limits_{n=1}^{\infty} f(n)$ converges" means that the sequence $(\underbrace{f(1) + \ldots + F(N)}_{=:S_N})$ has a limit as $N \to \infty$.

Since $f > 0$, the sequence (S_N) is increasing. So if (S_N) is bounded then (S_N) has a limit as $N \to \infty$, and if (S_N) is not bounded then (S_N) diverges to infinity as $N \to \infty$.

Because f is weakly decreasing, for all $x \in [n, n+1]$ we have

$$f(n) \geq f(x) \geq f(n+1).$$

Integrating over $[n, n+1]$, we obtain

$$f(n) \geq \int\limits_n^{n+1} f(x) \geq f(n+1).$$

Summing over $n = 1, \ldots, N$, we obtain

$$f(1) + f(2) + \ldots + f(N) \ \geq \ \int_1^{N+1} f \ \geq \ f(2) + \ldots + f(n) + f(n+1).$$

Namely, for all N we have

$$S_N \geq F(N+1) \geq S_{N+1} - f(1).$$

Note that $f(1)$ is a constant independent of N. From the second inequality we conclude that if the function F is bounded then the sequence (S_N) is bounded. From the first inequality, together with the fact that F is increasing and with the Archimedean property of the real numbers, we conclude that if the sequence (S_N) is bounded then the function F is bounded. $\square$

Theorem (Comparison test)**.** *Assume that for all n we have $0 \le a_n \le b_n$. Then if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Because a_n and b_n are all non-negative, the series $\sum b_n$ converges if and only if the sequence $\{\sum_{n=1}^{N} b_n\}_{N\in\mathbb{N}}$ is bounded, and the series $\sum a_n$ converges if and only if the sequence $\{\sum_{n=1}^{N} a_n\}_{N\in\mathbb{N}}$ is bounded. Because $a_n \le b_n$ for all n, if the sequence $\{\sum_{n=1}^{N} b_n\}_{N\in\mathbb{N}}$ is bounded, then the sequence $\{\sum_{n=1}^{N} a_n\}_{N\in\mathbb{N}}$ is bounded. $\square$

Theorem (Ratio test)**.** *Assume that $a_n > 0$ for all n and that $\dfrac{a_{n+1}}{a_n} \xrightarrow[n\to\infty]{} r$. If $r < 1$, then $\sum a_n$ converges. If $r > 1$, then $\sum a_n$ diverges.*

Proof. Assume $r < 1$. Let ρ be such that $r < \rho < 1$. Let N be such that for all $n \ge N$ we have $\dfrac{a_{n+1}}{a_n} < \rho$. (Why does such an N exist?) Then for all $k \ge 1$ we have

$$a_{N+k} = a_N \cdot \underbrace{\frac{a_{N+1}}{a_N} \cdot \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+k}}{a_{N+k-1}}}_{k \text{ terms}} \le a_N \rho^k.$$

Because $0 < \rho < 1$, the geometric series $\sum a_N \rho^k$ converges. By the comparison test, the series $\sum_{k=1}^{\infty} a_{N+k}$, and hence the series $\sum_{n=1}^{\infty} a_n$, also converges.

Now assume $r > 1$. Let N be such that for all $n \ge N$ we have $\dfrac{a_{n+1}}{a_n} > 1$. (Why does such an N exist?) Then for all $k \ge 1$ we have $a_{N+k} > a_N$. So the sequence $(a_n)_{n\ge N}$, being bounded from below by the positive number a_N, does not converge to 0; and so the series $\sum a_n$ does not converge to any limit. $\square$

Example: if $0 \le r < 1$ then $\sum nr^n$ converges by the ratio test, because the sequence of consecutive ratios $\frac{(n+1)r^{n+1}}{nr^n} = \dfrac{n+1}{n} r$ converges to r.

Example: let $\alpha > 0$, and consider the series $\sum_{n=1}^{\infty} \dfrac{1}{n^\alpha}$. The sequence of consecutive ratios $\frac{1/(n+1)^\alpha}{1/n^\alpha} = (n/n+1)^\alpha$ converges to 1, so the ratio test is inconclusive. But by the integral test, the series converges if $\alpha > 1$ and diverges to infinity if $0 < \alpha \le 1$. (And if $\alpha \le 0$ then the terms $1/n^\alpha$ do not converge to zero so the series doesn't converge.)

The previous criteria apply to series whose summands are positive. For more general series we can combine these criteria with the following theorem.

Theorem (Absolute convergence implies convergence)**.**

If $\sum_n |a_n|$ converges, then $\sum_n a_n$ converges.

The proof of this theorem relies on the Cauchy criterion for convergence.

Theorem (Cauchy criterion for series)**.** *A series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for every* $m > n > N$ *we have* $|a_{n+1} + \ldots + a_m| < \epsilon$.

Proof. Let $s_n = a_1 + \ldots + a_n$. By definition, the series $\sum a_n$ converges if and only if the sequence (s_n) converges. By the Cauchy criterion for a sequence, this holds if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $m > n > N$ we have $|s_m - s_n| < \epsilon$. Writing $s_m - s_n = a_{n+1} + \ldots + a_m$, we obtain the Cauchy criterion for the series. □

Proof that absolute convergence implies convergence. Suppose that the series $\sum_n |a_n|$ converges. Then it satisfies the Cauchy criterion. Let $\epsilon > 0$. Let N be such that for all $m > n > N$ we have $|a_{n+1} + \ldots + |a_m| < \epsilon$. By the triangle inequality, $|a_{n+1} + \ldots + a_m| \leq |a_{n+1}| + \ldots + |a_m|$, so $|a_{n+1} + \ldots + a_m|$ is also $< \epsilon$. So the series $\sum a_n$ also satisfies the Cauchy criterion; so the series $\sum a_n$ converges. □

Definition. A series $\sum a_n$ **converges absolutely** if the series $\sum |a_n|$ converges.

Thus, if a series converges absolutely, then the series converges.

Example: let $-1 < x < 1$. By the earlier example, the series $\sum n|x|^n$ converges. So $\sum nx^n$ (converges absolutely, hence) converges.

Example: the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ does not converge absolutely.

Does it converge? Yes, by the Leibniz criterion:

Theorem (Leibniz criterion)**.** *Let* (b_n) *be a weakly decreasing sequence of positive numbers that converges to zero. Then* $\sum (-1)^{n+1} b_n$ *converges.*

Proof. Let $s_n = b_1 - b_2 + b_3 - \ldots + (-1)^{n+1} b_n$. Then, for all k,

$$s_{2k} \leq s_{2k+2} \leq s_{2k+3} \leq s_{2k+1}.$$

(Please make sure that you see why.) So

$$\inf\{s_k\}_{k \geq N} = \begin{cases} s_N & \text{if } N \text{ is even} \\ s_{N+1} & \text{if } N \text{ is odd} \end{cases}$$

and

$$\sup\{s_k\}_{k \geq N} = \begin{cases} s_N & \text{if } N \text{ is odd} \\ s_{N+1} & \text{if } N \text{ is even} \end{cases} .$$

So $\underline{\lim}\, s_n = \lim(s_2, s_4, s_6, \ldots)$ and $\overline{\lim}\, s_n = \lim(s_1, s_3, s_5, \ldots)$. By algebraic properties of limits,

$$\overline{\lim}\, s_n - \underline{\lim}\, s_n = (\lim_{n\to\infty} s_{2n+1}) - (\lim_{n\to\infty} s_{2n}) = \lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} b_{2n+1} = 0.$$

So $\underline{\lim}(s_n) = \overline{\lim}(s_n)$, which implies that the sequence (s_n), and hence the series $\sum a_n$, converges. □