UNIFORM CONVERGENCE

MAT157, WINTER 2021. YAEL KARSHON

This is a second set of notes that supplement Spivak's Chapter 24.

The following three examples show that convergence of functions might not be "well behaved" with respect to continuity, integration, and differentiation.

(1) There is a sequence of continuous functions f_n that converges pointwise to a discontinuous function f. For example, take $f_n(x) = x^n$ for $0 \le x \le 1$ and

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(2) There is a sequence of Darboux integrable functions $f_n \colon [0,1] \to \mathbb{R}$ that converges pointwise to a Darboux integrable function $f \colon [0,1] \to \mathbb{R}$ but such that the sequence of numbers $\int_0^1 f_n$ does not converge to the number $\int_0^1 f$. For example, take

$$f_n(x) = \begin{cases} 4n^2 x & \text{if } 0 \le x \le \frac{1}{2n} \\ 4n - 4n^2 x & \text{if } \frac{1}{2n} < x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \le 1 \end{cases}$$

and $f \equiv 0$.

(3) There is a sequence of differentiable functions f_n that converges uniformly to a differentiable function f but such that the sequence of functions f_n' does not converge to the function f'. For example, take $f_n(x) = \frac{1}{n} \sin nx$ and $f \equiv 0$.

The following three theorems give conditions under which convergence of functions *is* "well behaved" with respect to continuity, integration, and differentiation.

Exercise. Suppose $f_n \to f$ uniformly on some set A. Then for each $\hat{\epsilon} > 0$ there exists an n such that $|f_n(x) - f(x)| < \hat{\epsilon}$ for all $x \in A$.

Theorem 1. *Suppose $f_n \to f$ uniformly on some neighbourhood U of a point γ. Assume that for each n the function f_n is continuous at γ. Then the function f is continuous at γ.*

Proof. For each n and each $x \in U$,

$$|f(x) - f(\gamma)| \;\le\; |f(x) - f_n(x)| \;+\; |f_n(x) - f_n(\gamma)| \;+\; |f_n(\gamma) - f(\gamma)|.$$

The middle term can be made arbitrarily small by requiring x to be close enough to γ. But how close is "close enough" depends on n. So we must first fix an n and only then decide how close x must be to γ.

Let $\epsilon > 0$. Let n be such that $|f_n(x) - f(x)| < \epsilon/3$ for all $x \in U$. Then the first and third summand above are $< \epsilon/3$. Let $\delta > 0$ be such that $|f_n(x) - f_n(\gamma)| < \epsilon/3$ for all x in the

δ-neighbourhood of γ. Then for all such x, the second summand above is also $< \epsilon/3$, and so $|f(x) - f(\gamma)| < \epsilon$. ☐

Theorem 2. *Suppose $f_n \to f$ uniformly on $[a, b]$. Suppose that for each f_n the function f_n is Darboux integrable on $[a, b]$. Then*

(i) The function f is Darboux integrable on $[a, b]$.

(ii) The sequence $\int_a^b f_n$ converges to $\int_a^b f$.

For the proof of (ii) assuming (i), see Spivak's book.

We now prove (i). Recall the criterion for integrability:

> A function $g\colon [a, b] \to \mathbb{R}$ is Darboux integrable if and only if for every $\epsilon > 0$ there exists a partition of $[a, b]$ such that $|U(f, P) - L(f, P)| < \epsilon$.

Let $\epsilon > 0$. We need to find a partition P such that $|U(f, P) - L(f, P)| < \epsilon$. Let $\widehat{\epsilon} = \frac{\epsilon}{1+2(b-a)}$. Let m be such that $|f(x) - f_m(x)| < \widehat{\epsilon}$ for all $x \in [a, b]$. For any partition P of $[a, b]$ and any interval $[t_{j-1}, t_j]$ of the partition, since $f_m - \widehat{\epsilon} < f < f_m + \widehat{\epsilon}$, we have

$$\sup_{[t_{j-1},t_j]} f_m \; - \; \widehat{\epsilon} \; \leq \; \sup_{[t_{j-1},t_j]} f \; \leq \; \sup_{[t_{j-1},t_j]} f_m \; + \; \widehat{\epsilon}.$$

Multiplying by $(t_j - t_{j-1})$ and summing over j, we obtain that

$$U(f_m, P) - \widehat{\epsilon}(b - a) \leq U(f, P) \leq U(f_m, P) + \widehat{\epsilon}(b - a).$$

A similar argument applies to the lower Darboux sums. So

$$|U(f, P) - U(f_m, P)| \; \leq \; \widehat{\epsilon}\,(b - a) \quad \text{and} \quad |L(f, P) - L(f_m, P)| \; \leq \; \widehat{\epsilon}\,(b - a).$$

All this works for any partition P. Now, let P be a partition of $[a, b]$ such that $|U(f_m, P) - L(f_m, P)| < \widehat{\epsilon}$. Then

$$\begin{aligned} &|U(f, P) - L(f, P)| \\ &\leq \underbrace{|U(f, P) - U(f_m, P)|}_{\leq \widehat{\epsilon}(b-a)} + \underbrace{|U(f_m, P) - L(f_m, P)|}_{< \widehat{\epsilon}} + \underbrace{|L(f_m, P) - L(f, P)|}_{\leq \widehat{\epsilon}(b-a)} \\ &< \epsilon. \end{aligned}$$

Theorem 3. *Let f_n for $n \in \mathbb{N}$, f, and g be functions on $[a, b]$. Suppose that f_n are C^1 (continuously differentiable), that $f_n \to f$ pointwise, and that $f_n' \to g$ uniformly. Then f is C^1, and $f' = g$.*

Proof. Because f_n are C^1, their derivatives f_n' are continuous, hence Darboux integrable. Because $f_n \to g$ uniformly, by the earlier theorem about continuity g is also continuous. By the earlier theorem about integrals applied to the intervals $[a, x]$,

$$\int_a^x f_n' \xrightarrow{n \to \infty} \int_a^x g.$$

But

$$\int_a^x f_n' = f_n(x) - f_n(a) \xrightarrow{n \to \infty} f(x) - f(a),$$

where the first equality is by the second fundamental theorem of calculus and the second equality is because $f_n \to f$ pointwise. So $\int_a^x g = f(x) - f(a)$, which we rewrite as

$$f(x) = f(a) + \int_a^x g.$$

By the first fundamental theorem of calculus, the right hand side is differentiable and its derivative is $g(x)$. It follows that the f is differentiable and its derivative is g. $\square$

The special case of power series is particularly nice.

Theorem. *Suppose that a power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges to a function* f *on the interval* $(-r, r)$*. Then*

(i) The function $f\colon (-r, r) \to \mathbb{R}$ *is* C^1*, and*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = f'(x).$$

(ii)

$$\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} = \int_0^x f.$$

Proof. Fix any $x_0 \in (-r, r)$. Choose any $\hat{r}$ such that $|x_0| < \hat{r} < r$. Because $\hat{r} \in (-r, r)$, the series $\sum_{n=0}^{\infty} \hat{r}^n$ converges. Fix a ρ such that $|x_0| < \rho < \hat{r}$. By a theorem from the previous handout, the three series

$$\sum_{n=0}^{\infty} a_n x^n \quad , \quad \sum_{n=1}^{\infty} n a_n x^{n-1} \quad , \quad \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

all converge uniformly on $[-\rho, \rho]$.

Restricting to a neighbourhood of x_0 and applying the earlier theorem about derivatives to the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n x^n$, we obtain (i).

Restricting to the interval $[0, x_0]$ (or to $[x_0, 0]$ if $x_0 < 0$) and applying the earlier theorem about integrals to the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n x^n$, we obtain (ii). $\square$

Corollary. *Suppose that* $\sum_{n=0}^{\infty} a_n (x - \gamma)^n = f(x)$ *near* γ*. Then* f *is differentiable near* γ*, and* $\sum_{n=1}^{\infty} n a_n (x - \gamma)^{n-1} = f'(x)$*.*

Proof. Apply the previous theorem to the series $\sum a_n \xi^n$, which converges to the function $f(\gamma + \xi)$. $\square$

Arguing inductively, we obtain expressions for the kth derivative of $\sum a_n (x - \gamma)^n$ for all k; substituting $x = \gamma$, we obtain the following corollary.

Corollary. *Suppose that* $\sum_{n=0}^{\infty} a_n (x - \gamma)^n = f(x)$ *near* γ*. Then* f *is differentiable to all orders at* γ*, and for all* n *we have* $a_n = \frac{1}{n!} f^{(n)}(\gamma)$*.*